Mathematical Aspects of Analytic Infinite

解析无穷大的数学问题

Derivative Gravity Theories

导数引力理论

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Abstract

摘要

In this note we would like to reiterate in a concise manner several issues of higher and infinite derivative field theories and gravity theories with a hope to wake an interest to solving these questions. Namely, (i) appearance of ghosts even if a ghost-free condition is fixed, (ii) background dependence of the presence of ghosts and possible cure with higher curvature terms, (iii) unexplored connection of the space of solutions in analytic infinite derivative gravity and existence of Fourier transform, and (iv) related problems.

在这篇短文中，我们旨在以简洁的方式重申高阶与无限导数场论、引力理论中的若干问题，希望能引发学界对解决这些问题的兴趣。这些问题分别是:(i) 即便固定了无鬼条件仍会出现鬼场；(ii) 鬼场存在的背景依赖性，以及高阶曲率项可能的解决作用；(iii) 解析无限导数引力中解空间与傅里叶变换存在性尚未探索的关联；(iv) 其他相关问题。

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Introduction and Motivation

引言与研究动机

This section serves as an introduction and motivation to the infinite derivative theories in general and infinite derivative gravity theories in particular. Several historical references are in order (We do not have a purpose to site here all the literature and silently assumes ”references therein” in most cases throughout this note. This applies both to the introductory part and to the more technical sections.). It was shown already by Ostrogradski in 1850, i.e., 170 years ago, that higher derivatives will lead to problems of energy unbounded from below [1]. Later on, in the first half of twentieth century, mathematical works appeared studying solutions of the infinite derivative equations with the motivation ”in case physicists will need them 1 day” [2-4].

What surprised me personally that a very comprehensive paper on the subject has appeared already in 1950 by Pais and Uhlembeck studying many of the field theoretical questions which we are going to discuss in this note [5].

本节对一般无限导数理论，尤其是无限导数引力理论进行介绍并阐述其研究动机。下文先列出若干历史文献 (本文无意在此罗列所有相关文献，多数情况下默认包含“其中引用的相关文献”，这一规则既适用于引言部分，也适用于更技术性的章节)。早在 1850 年，也就是 170 年前，奥斯特罗格拉德斯基就已经指出，更高阶导数会导致能量无下界的问题 [1]。之后，在 20 世纪上半叶，已有数学工作研究无限导数方程的解，当时的研究动机只是“说不定哪一天物理学家会用到它们”[2-4]。令我意外的是，早在 1950 年，佩斯和乌伦贝克就发表了一篇非常全面的论文，研究了许多本文将要讨论的场论相关问题 [5]。

Later, higher derivative and infinite derivative theories have started to arise more often in the literature with just some citations here [6-25] and notably in recent attempts of quantizing gravity [26-29]. However, one of the most solid appearances of such models happened in string field theory, the second quantized description of strings [30-33].

后来，高阶导数和无限导数理论越来越多地出现在文献中，此处仅列举部分文献 [6-25]，这类理论在近期的引力量子化尝试中尤为受关注 [26-29]。不过，这类模型最扎实的出处是弦场论，也就是弦的二次量子化描述 [30-33]。

We immediately start with an example. Naturally, before jumping onto gravity, one will benefit discussing a simple scalar field leaving gravity for the next sections. A model example providing such a higher derivative modification can be readily written as follows:

我们直接从一个例子开始。显然，在讨论引力之前，先分析一个简单的标量场会更有助于理解，引力我们留到后续章节讨论。这种高阶导数修正的模型示例可以直接写成如下形式:

$$S = \int d^4x \left[\frac{1}{2}\phi\left(\Box - m^2\right) f(\Box)^{-1}\phi - V(\phi)\right] \tag{1}$$

where $\Box$ is the d’ Alembertian operator. This toy example can be traced to string field theory which on top of many attractive features contains analytic infinite derivative (AID) operators or form factors acting on space-time fields. These operators appear analytic at low energies making IR limit easy to discuss. The characteristic scale, the so-called scale of non-locality $\mathcal{M}$ separates local (IR) and non-local (UV) regimes. Usually we set it to unity, but in the full picture we should have $f\left(\Box/\mathcal{M}^2\right)$.

其中 $\Box$ 是达朗贝尔算符。这个玩具模型可以追溯到弦场论，弦场论除了诸多吸引人的性质之外，还包含作用在时空场上的解析无限导数 (AID) 算符或称形状因子。这些算符在低能下是解析的，便于讨论红外极限。特征尺度即所谓的非定域性尺度 $\mathcal{M}$ 区分了定域 (红外) 和非定域 (紫外) 区域。通常我们将其设为单位 1，但完整形式下我们应当得到 $f\left(\Box/\mathcal{M}^2\right)$。

In this simple example, we may have a propagator of an infinite order in derivatives, and there is an interaction term which we assume generates a bounded from below potential. These are the features typical for non-local models that have applications to quantum gravity (renormalizable and ghost-free AID theories) [23] and UV finite non-local scalar theories with an arbitrary potential [24, 34, 35] . For the metric signature

$(-+++)$, the above Lagrangian describes a normal non-ghost field if $f=1$ which is moreover not a tachyon for $m^2>0$.

在这个简单例子中，我们可以得到导数阶数无穷的传播子，还有一个相互作用项，我们假设它能生成一个下有界的势能。这些都是非定域模型的典型特征，这类模型可应用于量子引力 (可重整且无鬼的解析无限导数理论)[23]，以及带任意势 [24, 34, 35] 的紫外有限非定域标量理论。对度规符号 $(-+++)$ 而言，若满足 $f=1$ ，上述拉格朗日量描述的就是一个正常的无鬼场，且当 $m^2>0$ 时，它也不是快子。

Namely the ability of higher derivatives to make theories known before as non-renormalizable UV complete makes such constructions extremely attractive.

正是高阶导数能让原本不可重整的理论在紫外完备，这一特性让这类构造极具吸引力。

To see why SFT plays the role of the most robust motivation for theories with an infinite number of higher derivatives in the form of analytic form factors, let's see the kind of Lagrangians which arise in the SFT construction [32,33]:

为了说明弦场论为何能成为解析形状因子形式的无穷高阶导数理论最可靠的研究动机，我们来看弦场论构造中得到的拉格朗日量形式 [32,33]:

$$L_{\mathrm{SFT}}=\frac{1}{2}\varphi\left(\Box-m^2\right)\varphi-V\left(e^{-\sigma(\Box)}\varphi\right), \tag{2}$$

where $\sigma(\Box)$ is some polynomial of the d' Alembertian $\Box$ and V is some interaction, polynomial in its argument and the lowest degree of the field φ in V is 3 . In other words, V does not contain quadratic in φ terms. Technically V is not an interaction potential anymore since it has clear momentum dependence. In string theory, the scale of non-locality is the string mass which is theoretically bounded by the Planck mass M_P from above.

其中 $\sigma(\Box)$ 是达朗贝尔算符 $\Box$ 的某个多项式，V 是某个相互作用项，是其自变量的多项式，场 φ 在 V 中的最低次数为 3。换句话说，V 不包含 φ 的二次项。从技术上讲，V 已经不再是相互作用势，因为它有明确的动量依赖。在弦论中，非定域性尺度是弦质量，理论上它的上界为普朗克质量 M_P 。

From the perspective of the latter Lagrangian, we have a theory with AID vertices. One can however easily redefine the field as $\phi=e^{-\sigma(\Box)}\varphi$ and move the AID operators to the quadratic in fields term. This yields:

从上述拉格朗日量的视角来看，我们得到了一个带解析无限导数顶点的理论。不过我们可以很容易地将场重新定义为 $\phi=e^{-\sigma(\Box)}\varphi$ ，把解析无限导数算符移到场的二次项中，得到:

$$L=\frac{1}{2}\phi\left(\Box-m^2\right)e^{2\sigma(\Box)}\phi-V(\phi), \tag{3}$$

which is exactly Lagrangian (1) where $f(\Box)^{-1}=e^{2\sigma(\Box)}$.

这正好就是拉格朗日量 (1)，其中 $f(\Box)^{-1}=e^{2\sigma(\Box)}$ 。

In a well-defined scenario, one should avoid ghost fields, and this is one of the guiding principles limiting our AID theory construction. We know starting from the paper by Ostrogradski [1] that higher derivatives generically introduce ghosts. Since the number of degrees of freedom is counted by the number of poles in the propagator, one may try to keep only one pole even with extra derivatives in the Lagrangian. Given the construction above, this can be made only if the original operator $\left(\Box - m^2\right)$ in the quadratic form is multiplied by a function of the d'Alebertian which has no zeros on the whole complex plane. Mathematically the only possibility for such an extra factor is an exponent of an entire function. This being said as long as the ghost absence question is concerned, we can consider $\sigma(\Box)$ to be a generic entire function and not only a polynomial. The presented construction can be trivially elevated to models in a curved background by a simple replacement of the flat space-time d'Alembertian by its covariant counterpart. This will not spoil the ghost-free condition in any way.

在定义明确的框架中，应当避免鬼场，这是限制我们构建无限导数 (AID) 理论的指导原则之一。我们知道，从奥斯特罗格拉德斯基的论文 [1] 开始，高阶导数通常会引入鬼场。由于自由度数量由传播子的极点数量决定，即便拉格朗日量中存在额外导数，我们也可以尝试只保留一个极点。根据上述构建过程，只有当二次型中的原算子 $\left(\Box - m^2\right)$ 乘上一个在整个复平面上都没有零点的达朗贝尔算符函数时，才能实现这一点。从数学上讲，这类额外因子唯一可能的形式是整函数的指数。因此，就无鬼要求而言，我们可以将 $\sigma(\Box)$ 视为一般的整函数，而非仅局限于多项式。上述构造可以平凡地推广到弯曲背景下的模型，只需将平坦时空的达朗贝尔算符替换为协变形式的达朗贝尔算符即可。这种替换完全不会破坏无鬼条件。

Higher and Infinite Derivative Theories and Ghosts

高阶导数与无限导数理论及鬼场

We stress however that the non-local function itself depends on the particular vacuum of the potential in order to guarantee the absence of ghosts. Namely, consider a potential that has several vacua which moreover arrange different masses to the scalar field. The presented above construction allows having no ghosts only in a single vacuum in which the mass of the field is given by m^2 . In other words, in this vacuum $V''(\phi) = 0$.

但我们需要强调，非定域函数本身依赖于势的特定真空，以此保证不存在鬼场。也就是说，考虑一个存在多个真空的势，不同真空会给标量场赋予不同的质量。上述构造仅在单一真空中能保证无鬼场，该真空中场的质量由 m^2 给出。换句话说，在该真空中 $V''(\phi) = 0$ 。

As long as $V'' \neq 0$ in a vacuum which is true in a generic situation or around a classical background, an attempt to linearize equations results in the following operator in the quadratic form:

只要 $V'' \neq 0$ 处于一般情况成立的真空或经典背景附近，对方程线性化后，二次型中会得到如下算子:

$$\mathcal{K}_m(\Box) = \frac{1}{2}\left(\Box - m^2\right)e^{2\sigma(\Box)} - \frac{1}{2}m^2. \tag{4}$$

Here we use $m^2 = V''(\phi)$ with ϕ being the classical background value of the scalar field or simply a nontrivial vacuum. This particular operator results in infinitely many effective local scalar fields with most of them having complex masses squared.

此处我们使用 $m^2 = V''(\phi)$，其中 ϕ 是标量场的经典背景值，或简单说是非平凡真空。该特殊算子对应无穷多个有效局域标量场，其中大多数有效场的质量平方为复数。

Suppose we consider a theory in which in all vacua $V''(\phi) = 0$. Then we name these new fields effective because they do not belong to any vacuum of the model in a sense that they cannot be created as states in any present vacuum of the theory. Still, it is nothing wrong with linearizing around a non-vacuum point of evolution especially if the effective mass is a slowly varying quantity as it is for any nearly flat potential (e.g., during inflation).

假设我们考虑一个所有真空中都满足 $V''(\phi) = 0$ 的理论。我们将这些新场称为有效场，因为它们不属于模型的任何真空——即它们无法在该理论的任意现有真空中作为态被产生。即便如此，在演化的非真空点做线性化并没有问题，尤其是当有效质量变化缓慢时，任何缓变势都满足这一情况 (例如暴胀过程中)。

To understand why many fields appear, it is useful to represent the operator function of the d'Alembertian utilizing the Weierstrass product decomposition for an entire function which tells us that any entire function $\mathcal{G}(z)$ can be presented as

为理解为什么会出现多个场，我们可以利用整函数的魏尔斯特拉斯乘积分解来表示达朗贝尔算符的算子函数，该分解指出任意整函数 $\mathcal{G}(z)$ 都可以写成

$$\mathcal{G}(z) = \prod_i (z - z_i)^{n_i} e^{g(z)}, \tag{5}$$

where z_i are algebraic roots of the equation $\mathcal{G}(z) = 0, n_i$ is the root' s multiplicity and $g(z)$ is some entire function. In a simplest case, all $n_i = 1$. The outlined above operator $\mathcal{K}$ is a manifestly entire function. A further justification to make use of this formula in general comes from the fact that at least the string field theory obtained models certainly contain only entire functions of $\square$ as operators in field quadratic forms. Therefore, in a generic case for models containing the following quadratic in field term

其中 z_i 是方程的代数根，$\mathcal{G}(z) = 0, n_i$ 是根的重数，$g(z)$ 是某个整函数。最简单的情况下，所有 $n_i = 1$ 。上述给出的算子 $\mathcal{K}$ 显然是一个整函数。一般情况下使用该公式的进一步依据是，至少弦场论得到的模型中，场二次型的算子确实只包含关于 $\square$ 的整函数。因此，对于包含如下场二次项的一般模型

$$L = \frac{1}{2}\phi\mathcal{G}(\square)\phi + \dots, \tag{6}$$

with an entire operator function $\mathcal{G}(\square)$, one can easily achieve two things [36, 37] :

对于整算子函数 $\mathcal{G}(\square)$ ，我们可以很容易得到两个结果 [36, 37] :

- First, the free equation of motion can be easily solved. Indeed, it would look like

- 首先，自由运动方程可以很容易求解。具体来说，方程形式为

$$\mathcal{G}(\Box)\phi = \prod_i \left(\Box - m_i^2\right) e^{g(\Box)}\phi = 0. \tag{7}$$

We assume here for simplicity that the root's multiplicity is always one. Then the solution will be

此处为简化我们假设所有根的重数都为 1，那么解为

$$\phi = \sum_i \phi_i \text{ where } \left(\Box - m_i^2\right)\phi_i = 0. \tag{8}$$

Moreover, since the original model provides a form factor with real coefficients in its Taylor expansion around zero, all roots are either real or come in complex conjugate pairs.

此外，由于原模型在零点泰勒展开给出的形状因子系数均为实数，因此所有根要么是实数，要么以共轭复根对的形式出现。

For a pair of complex conjugate roots, numbered say i and j , one should consider correlated initial conditions on functions ϕ_i and ϕ_j such that the sum of these functions is zero. This is a condition for a consistent background solution for the original field ϕ which must be real.

对于一对共轭复根，例如编号为 i 和 j 的根，需要对函数 ϕ_i 和 ϕ_j 施加关联初始条件，使得两函数之和为零。这是保证原场 ϕ 的背景解自洽且为实数的必要条件。

We note here that more than one real m_i^2 will definitely be a ghostlike excitation and is as such a problematic configuration. However, pairs of complex conjugate masses squared are not obligatory bad and may lead to hassle-free models. Notice that in this case, one encounters somewhat noncanonical complex field model which has the following Lagrangian:

我们在此指出，多个实 m_i^2 必然对应类鬼激发，属于有问题的构型。但共轭复质量平方对并不一定是坏的，反而可以得到无问题的模型。注意在这种情况下，我们得到的是非标准的复场模型，其拉格朗日量为:

$$L_i = \frac{1}{2}\left[\mathcal{G}'\left(m_i^2\right)\phi\left(\Box - m_i^2\right)\phi + \mathcal{G}'\left(m_i^{2^*}\right)\phi^*\left(\Box - m_i^{2^*}\right)\phi^*\right], \tag{9}$$

which cannot be diagonalized in real fields. Factors of $\mathcal{G}'\left(m_i^2\right)$ appear upon computing the Weierstrass decomposition.

它无法在实场空间对角化。计算魏尔斯特拉斯分解时会得到 $\mathcal{G}'\left(m_i^2\right)$ 因子。

The appearance of complex conjugate poles and their interpretation was already discussed in [7]. In a nutshell, such poles with masses $m = u \pm iv$ with $v > 0$ would imply causality violation at distances less than $1/\sqrt{v}$. Even generically an alarming symptom, it can be safely ignored given that the imaginary part which would cause troubles is large enough compared to physically important scales of the model. On top of this, absence of classical growing modes should be guaranteed to claim safely that newly appeared particles do not interfere with the rest of the model. It is important to verify that this wishful expectation holds.

复共轭极点的出现及其解释已在文献 [7] 中讨论过。简言之，这类质量为 $m = u \pm iv$ 、满足 $v > 0$ 的极点会在小于 $1/\sqrt{v}$ 的距离上违反因果性。虽然这通常是一个值得警惕的问题，但只要引发问题的虚部远大于模型中所有物理重要尺度，就可以放心将其忽略。除此之外，要确保新出现的粒子不会干扰模型的其他部分，还必须保证不存在增长的经典模式。验证这一预期是否成立十分重要。

- Second, one can straightforwardly compute residues at all poles of the propagator. The point is to figure out by use of formula (5) that a residue at the point of m_i^2 is given by $1/\mathcal{G}'\left(m_i^2\right)$ which is not zero by construction as long as all roots are assumed to be the simple ones.

- 其次，我们可以直接计算传播子所有极点处的留数。核心是通过公式 (5) 得到 m_i^2 点处的留数为 $1/\mathcal{G}'\left(m_i^2\right)$ ，只要假设所有根都是单根，根据构造该留数就不为零。

One technically important point here is that in principle fields can be rescaled to a somewhat arbitrary number. This implies that unless we clearly understand the field's normalization, or unless there are no eternal guiding principles for doing that, one can always bring the real part of the residue value to be ± 1 .

此处一个技术上的重点是，场原则上可以按任意倍数重新标度，这意味着除非我们明确了解场的归一化，或是存在确定归一化的统一指导原则，否则总可以将留数的实部调整为 ± 1 。

The novel and an intuition breaking thing here is that the very situation of changing the number of degrees of freedom dynamically is extremely unusual and is met here only due to higher derivatives. Moreover, the jump is bizarre from a single excitation to effectively infinitely many of them, and most of those extra "would-be" excitations have complex masses which are totally alien objects in canonical field theory. What is even more curious, neither vacuum of the presented theory can create anything but one real massless particle.

此处新颖且打破直觉的点在于: 动态改变自由度数量这种情况本身就极其反常，仅在高阶导数理论中才会出现。此外，自由度从单个激发跳变到实际上无穷多个，这十分怪异；而且这些额外的“准”激发大多具有复质量，这在正则场论中是完全异类的存在。更有意思的是，本文提出的理论中，任何真空都只能产生一个真实的无质量粒子。

Hence, in principle, we can say that those effective fields are not excitations in any way and as such just stop discussing them. However, since their appearance, even effective, is not a completely understood process, we are going to follow a safer way and show that these "would-be" degrees of freedom are screened by having huge masses compared to real physical scales in the model. This follows from the adjustment of the non-locality scale Λ to be heavier than the Hubble scale during inflation. Also we will formulate a condition allowing no classical growing modes for these new modes.

因此原则上我们可以认为这些有效场根本不是激发，直接不讨论它们即可。但由于它们的出现 (即使是有效出现) 仍是一个未被完全理解的过程，我们将采取更稳妥的思路: 证明这些“准”自由度由于质量远大于模型中的真实物理尺度而被屏蔽。这一结论源于我们将非定域性尺度 Λ 调整为比暴胀时期的哈勃尺度更重。我们还将给出这些新模式不存在经典增长模式的条件。

Masses of effective particles are given by roots of an algebraic equation [36]:

有效粒子的质量由下述代数方程的根给出 [36]:

$$\mathcal{K}_m\left(m_k^2\right) = 0 \tag{10}$$

What happens is that the masses of the fields are very large compared to the non-locality scale. This is however not enough to be relaxed about their presence, given that half of them are ghosts.

实际情况是，这些场的质量相比非定域性尺度已经非常大。但这还不足以让我们对它们的存在放下心来，因为其中一半都是鬼场。

Additionally we want to see that it is possible to have no growing classical solutions for these new effective modes. This simply boils down to solving the free equations of motion of the form:

此外我们还希望确认，这些新的有效模式确实可以不存在增长的经典解。这一问题可以简化为求解如下形式的自由运动方程:

$$\left(\Box - m_k^2\right)\phi = 0 \tag{11}$$

for all modes enumerated by k .

对所有由 k 标记的模式成立。

Let's start with the background d'Alembertian operator evaluated on the de Sitter background. Even though the equation looks familiar, it gets a new twist because m_k^2 is complex. Given that the Hubble parameter is denoted as H for our background de Sitter space-time, the solution to the latter equation is given by

我们先从德西特背景下的达朗贝尔算符开始讨论。虽然这个方程看起来很熟悉，但它出现了新的变化，因为 m_k^2 是复数。我们用 H 表示背景德西特时空的哈勃参数，上述方程的解为

$$\phi = e^{-\frac{3}{2}Ht}\left(\alpha J_\rho\left(\frac{ka_0}{H}e^{-Ht}\right) + \beta Y_\rho\left(\frac{ka_0}{H}e^{-Ht}\right)\right) \text{ with } \rho = -\sqrt{\frac{9}{4} - \frac{m_k^2}{H}}$$

(12)

where J_ρ, Y_ρ are Bessel functions of the first and second kind and a_0 is the normalization of the scale factor in the metric tensor at $t = 0$ and α, β are integration constants. Absence of growing solutions means that for large times t which correspond to small arguments of the Bessel functions both branches with coefficients α and β at most freeze to constants or have nongrowing oscillations. This results in the demand that both functions in the solution grow at most as $e^{3Ht/2}$. Series expansion of Bessel functions with index ρ near the origin tells:

其中 J_ρ, Y_ρ 为第一类和第二类贝塞尔函数，a_0 是度规张量中尺度因子在 $t = 0$ 处的归一化，α, β 为积分常数。不存在增长解意味着，对于对应贝塞尔函数小宗量的大时间 t ，系数分别为 α 和 β 的两个分支最多冻结为常数，或呈现非增长振荡。这要求解中的两个函数增长速度最多不超过 $e^{3Ht/2}$ 。原点附近指标为 ρ 的贝塞尔函数级数展开为:

$$J_\rho(x) \sim x^\rho, Y_\rho \sim x^{-\rho} \tag{13}$$

and it follows from here that we are good to go if $\mathrm{Re}\,(\rho) \leq 3H/2$. Upon some algebra one can figure out that this corresponds to

由此可知，当满足条件 $\mathrm{Re}\,(\rho) \leq 3H/2$ 时，我们的结论成立。经过代数推导可以得到该条件对应于

$$\left(\mathrm{Im}\left(m_k^2\right)\right)^2 < 9H^2\,\mathrm{Re}\left(m_k^2\right) \tag{14}$$

This selects the interior of a parabola-shaped domain on the complex plane. All solutions in (10) must satisfy this condition. This condition prompts for a careful choice of an entire function in the exponent of the infinite derivative operator, and we see that a simple choice of a polynomial does not fulfill the formulated requirement. However, known facts in the complex analysis do not impose any restriction to have such a function, and one can try to obtain the desired behavior by combining the Cauchy integral representation for holomorphic functions and the Weierstrass decomposition valid for entire functions. In particular one can deduce the following sufficient condition on the entire function in the exponent of the infinite derivative operator: the absolute value of this function should grow to infinity only along the positive real ray and be bounded in any other direction.

这就选出了复平面上一个抛物线形区域的内部。(10) 中的所有解都必须满足这一条件。该条件要求我们谨慎选择无限导数算子指数中的整函数，可见简单选择多项式无法满足已提出的要求。不过，复分析中的已有结论并未对存在这类函数施加任何限制，我们可以尝试结合全纯函数的柯西积分表示与适用于整函数的魏尔斯特拉斯分解来得到预期的行为。特别地，我们可以推导出无限导数算子指数中的整函数满足如下充分条件: 该函数的绝对值仅沿正实射线增长至无穷，且在所有其他方向上有界。

Turning to a usually more simple Minkowski background, we see that there is no way out of ghosts. H effectively goes to zero and solutions simply become:

转而讨论通常更简单的闵氏背景，我们发现无法摆脱鬼场。H 实际上趋于零，解可直接写为:

$$\phi = \phi_{0+}e^{im_k t} + \phi_{0-}e^{-im_k t} \tag{15}$$

meaning that as long as there is an imaginary part in m_k , one readily has exponentially growing solutions. One can boldly say that non-locality forbids pure Minkowski space and advocates any nonzero cosmological constant.

这意味着只要 m_k 中存在虚部，就会立刻得到指数增长的解。我们可以明确地说，非局域性不允许存在纯闵可夫斯基空间，支持非零宇宙学常数。

An interpretation of these new fields is still unclear, while there are several including reasonably old studies [5, 7, 38, 39] claiming that they do not spoil unitarity at least at scales below the scale of higher derivative modification $\mathcal{M}$. Another approach designates such fields as totally virtual degrees of freedom, and a term fakeon is used with respect to such excitations [16] especially in describing certain aspects of Lee-Wick-type models.

这些新场的物理解释目前仍不明确，但已有包括一些相当早期的研究 [5, 7, 38, 39] 声称，至少在高于高阶导数修正的尺度以下 $\mathcal{M}$ ，它们不会破坏幺正性。另一种观点将这类场归为完全虚的自由度，这类激发被称为赝场 fakeon[16]，该概念尤其常用于描述李-维克型模型的相关性质。

What is important to the currently ongoing study of the infinite derivative gravity [21], this problem is there as well. Let us recall the logic of the non-local gravity: adding higher curvature terms to the Einstein gravity action is not forbidden but generically creates ghosts; however, due to the success of the findings by Stelle, we see that the curvature-squared gravity is renormalizable; it is however non-unitary due to a ghost, and an attempt to cure the issue adding any arbitrary terms shows that only infinite number of derivatives can be combined in a ghost-free Lagrangian.

对于当前正在研究的无限导数引力 [21]，重要的是这个问题在该理论中同样存在。我们来回顾非局域引力的逻辑: 在爱因斯坦引力作用量中添加高阶曲率项并不被禁止，但一般会产生鬼场；不过，得益于 Stelle 的研究成果，我们知道曲率平方引力是可重整化的；但它因为存在鬼场而非幺正的，尝试添加任意项来解决该问题后发现，只有无穷多阶导数的组合才能构成无鬼场的拉格朗日量。

This section is important to understand how the same very problem backfires in higher derivative gravity. This general problem gets a new turn. Suppose we fix our gravitational Lagrangian, assuming it is a fundamental theory. Then only a single chosen gravitational background can be made ghost-free. Even though we just have seen that one can contain ghosts in some sense around de Sitter space, other backgrounds were not studied yet.

本节对于理解同一个问题如何在高阶导数引力中反覆出现十分重要。这个一般性问题出现了新的变化。假设我们确定了引力拉格朗日量，将其视为基本理论，那么只有一个选定的引力背景可以做到无鬼场。尽管我们已经看到，在德西特空间周围可以在某种意义上容纳鬼场，但其他背景尚未得到研究。

As one of the idea of one can try to solve the Klein-Gordon equation with a complex eigenvalue in a background of choice and see whether there are conditions how to bound all the solutions. This is obviously not an easy task though as in most situations one will not be able to get analytic solutions. A resolution to the moment is to leave this question for future study.

有一种思路是，我们可以尝试在选定背景下求解复本征值的克莱因-戈登方程，看看是否存在能约束所有解的条件。不过这显然不是一件易事，因为大多数情况下我们无法得到解析解。到目前为止，该问题只能留待未来研究。

One can wonder isn’t this enough to disregard infinite derivative theories as such? Our answer is ”no.” It is not enough. The present understanding of many theories leads to higher derivatives. There are plenty of unexplored yet ways to have a fresh look on them. At first we notice that SFT does have infinite derivatives and does have unitarity. This implies that our view on infinite derivative theories in the coordinate space-time formulation can be just incomplete. The other argument is that only infinite derivatives can make gravity ghost-free as long as diffeomorphism invariance and general Lorenz covariance are preserved, at least in the framework of Riemannian geometry.

有人可能会疑问，这还不足以直接否定整个无限导数理论吗？我们的答案是否定的，这并不够。目前对许多理论的认知都会导出高阶导数。仍然有大量尚未探索的路径，可以让我们重新审视这些理论。首先我们注意到，弦场论 SFT 确实存在无限导数，也确实保有幺正性。这意味着我们对坐标时空表述下无限导数理论的认知可能并不完整。另一个论据是，至少在黎曼几何框架内，只要保留微分同胚不变性和一般洛伦兹协变性，就只有无限导数能够让引力摆脱鬼场。

These aspects are going to be highlighted below.

这些方面将在下文展开讨论。

Most General AID Quadratic Gravity Action Around Maximally Symmetric Space-Times

最大对称时空背景下最一般 AID 二次引力作用量

Now we turn to gravity, namely, to quadratic in curvature gravity theory as it was shown some time ago that such a model is the most general approximation to study linear perturbations around maximally symmetric spacetimes (MSS). This section will come to the necessity of infinite derivative theories of gravity without any strings attached.

现在我们转向引力，具体来说是曲率二次项引力理论，此前已有研究表明，该模型是研究最大对称时空 (MSS) 线性微扰的最一般近似。本节将阐释无附加条件的无限导数引力理论的必要性。

Reduction of the Equivalent Action

等效作用量约化

One of the most crucial results of [40] provides a most generic action for studying linear perturbations around MSS. Consider the following action:

文献 [40] 最重要的结论之一，给出了研究最大对称空间 (MSS) 周围线性微扰的最一般作用量。考虑如下作用量:

$$S = \int d^4x\sqrt{-g}\left[P_0 + \sum_i \prod_I P_i O_{iI} Q_{iI}\right], \tag{16}$$

where P, Q depend only on the metric, Riemann tensor, and curvatures while O depend only on covariant derivatives. This action accommodates virtually all higher derivative gravity theories with an analytic dependence on curvatures and covariant derivatives. Assuming an existence of at least one (A)dS solution, the action relevant to study linear perturbations of EOM (coming from the quadratic variation of the action) around such a solution boils down to

其中 P, Q 仅依赖于度规、黎曼张量和曲率，而 O 仅依赖于协变导数。该作用量几乎涵盖了所有对曲率和协变导数存在解析依赖的高阶导数引力理论。假设至少存在一个 (反) 德西特解，研究该解周围运动方程 (EOM) 线性微扰的相关作用量 (由作用量的二阶变分得到) 可约化为

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R \right. \tag{17}$$

$$\left. + \frac{\lambda}{2}\left(R\mathcal{F}_1(\Box)R + R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu} + R_{\mu\nu\rho\sigma}\mathcal{F}_4(\Box)R^{\mu\nu\rho\sigma}\right) - \Lambda\right],$$

where λ is a dimensionless constant which is convenient to control the magnitude of the R^2 modification and Λ is an in principle possible cosmological constant term. Briefly the reduction is done by carefully accounting all possible terms which may contribute nontrivially to the second variation around MSS (and dropping all other terms). The fact which is heavily used on this way is that all curvature tensors on MSS are annihilated by covariant derivatives.

其中 λ 是一个无量纲常数，用于方便控制 R^2 修改项的幅度，Λ 原则上是一个可能的宇宙学常数项。该约化通过仔细计算所有可能对最大对称空间 (MSS) 周围二阶变分有非平庸贡献的项 (并丢弃所有其他项) 完成。在此过程中被大量使用的结论是: 最大对称空间 (MSS) 上所有曲率张量都被协变导数湮灭。

An important assumption essential for the actual computations and which was discussed in the Introduction is that all functions $\mathcal{F}$ are analytic. To be precise we need at the moment to have these functions analytic around zero. This is indeed required from the physical point of view. We want functions $\mathcal{F}(\Box)$ reduce to constants or vanish in a low-energy limit because we have to restore GR at very low energies. There is also other way to understand this. In writing $\mathcal{F}(\Box)$, we always assume that there is an energy scale of the gravity modification $\mathcal{M}$, which we name the scale of non-locality as in principle we may have infinite derivative operators ($\mathcal{M}$ should not be mixed with the much lower energy scale M at which the $R^2/6M^2$ term in the local R^2 inflationary model becomes comparable to the GR term R). This scale enters as $\mathcal{F}\left(\Box/\mathcal{M}^2\right)$. Even though for most of our technical steps we can put $\mathcal{M} = 1$, we still want to have a local or trivial limit once $\mathcal{M} \to \infty$ in order to be able to eventually restore GR. Hence, we come to the conclusion that functions $\mathcal{F}$ must be analytic at least in the origin.

实际计算中必不可少的一个重要假设 (已在引言中讨论过) 是: 所有函数 $\mathcal{F}$ 都是解析的。准确来说，我们目前要求这些函数在零点附近解析。从物理角度来看这确实是必要的。我们要求函数 $\mathcal{F}(\Box)$ 在低能极限下退化为常数或等于零，因为我们需要在极低能下恢复广义相对论 (GR)。还可以从另一个角度理解: 在写出 $\mathcal{F}(\Box)$ 时，我们始终假设引力修改存在一个能标 $\mathcal{M}$, 我们称之为非局域性标度，因为原则上我们可以拥有无穷阶导数算符 ($\mathcal{M}$ 不应与能量低得多的能标 M 混淆，在 M 处，局域 R^2 暴涨模型中的 $R^2/6M^2$ 项变得与 GR 项 R 相当)。该能标以 $\mathcal{F}\left(\Box/\mathcal{M}^2\right)$ 的形式进入作用量。尽管我们在大部分技术步骤中可以令 $\mathcal{M} = 1$ ，但我们仍然希望当 $\mathcal{M} \to \infty$ 时得到局域或平庸极限，以便最终恢复广义相对论。因此我们得出结论: 函数 $\mathcal{F}$ 至少在原点处必须解析。

Proposition. Action (17) is redundant in describing linear fluctuations around MSS.

命题: 作用量 (17) 在描述最大对称空间 (MSS) 周围线性涨落时是冗余的。

This proposition can be proven to be true because the previous analysis did not make use of Bianchi identities which is the cornerstone of the succeeding further reduction. To start with, action (17) can be rewritten as

该命题可被证明成立，因为此前的分析并未利用比安基恒等式，而比安基恒等式是后续进一步约化的基石。首先，作用量 (17) 可以改写为

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R \right. \tag{18}$$

$$\left. +\frac{\lambda}{2}\left(R\widetilde{\mathcal{F}}_R(\Box)R + L_{\mu\nu}\mathcal{F}_L(\Box)L^{\mu\nu} + W_{\mu\nu\rho\sigma}\widetilde{\mathcal{F}}_W(\Box)W^{\mu\nu\rho\sigma}\right) - \Lambda\right].$$

The purpose of using the Weyl tensor W and L -tensor which is the traceless Ricci tensor $L_{\mu\nu} = R_{\mu\nu} - \frac{1}{D}Rg_{\mu\nu}$ is their simplicity. Both are identically zero on MSS. Moreover, W is zero on any conformally flat background. The term to be attacked by Bianchi identities is the L -piece. A good reason to tackle this term somehow is that being simple on the background, it produces tremendous complications while trying to compute perturbations.

使用外尔张量 W 和无迹里奇张量 $L_{\mu\nu} = R_{\mu\nu} - \frac{1}{D}Rg_{\mu\nu}$ 即 L 张量的好处在于它们形式简单: 二者在最大对称空间 (MSS) 上都恒等于零。此外，W 在任意共形平坦背景上都等于零。将被比安基恒等式处理的项是 L 部分。必须处理这项的一个合理原因是: 尽管它在背景上形式简单，但在计算微扰时会带来极大的复杂性。

To make the long story short, we put aside all the technical details of the reduction. Upon a lengthy but a straightforward procedure, the full resulting action relevant for the study of linear perturbations around MSS vacua of (16) becomes:

长话短说，我们在此略过约化的所有技术细节。经过一套冗长但直接的计算流程，用于研究 (16) 的最大对称空间 (MSS) 真空周围线性微扰的完整最终作用量为:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + \frac{\lambda}{2}\left(R\mathcal{F}_R(\Box)R + W_{\mu\nu\rho\sigma}\mathcal{F}_W(\Box)W^{\mu\nu\rho\sigma}\right) - \Lambda\right]. \tag{19}$$

We consider this action as a significant simplification of (18) for several reasons (We note that our derivation is almost dimension independent. The only local term which survives in higher dimensions is the local square of L -tensor which we can drop in $D = 4$ due to the presence of the Gauss-Bonnet invariant. As such, the full action relevant for study of linear perturbations around MSS vacua of (16) formulated in $D > 4$ can be written as follows:

我们认为该作用量相较 (18) 有显著简化，原因如下 (我们注意到我们的推导几乎不依赖维度: 高维中唯一留存的局域项是 L 张量的局域平方，由于高斯-博内不变量的存在，我们可以在 $D = 4$ 中丢弃该项。因此，定义在 $D > 4$ 上、用于研究 (16) 的最大对称空间 (MSS) 真空周围线性微扰的完整作用量可写为:

$$S = \int d^Dx\sqrt{-g}\left[\frac{M_P^2}{2}R + \frac{\lambda}{2}\left(R\mathcal{F}_R(\Box)R + L_{\mu\nu}^2 + W_{\mu\nu\rho\sigma}\mathcal{F}_W(\Box)W^{\mu\nu\rho\sigma}\right) - \Lambda\right].$$

(20)

We still consider using L -tensor is preferred as it is identically zero on MSS.):

我们仍然认为使用 L 张量更优，因为它在最大对称空间 (MSS) 上恒等于零。):

(i) it contains only Ricci scalar and Weyl tensor and no Ricci tensor or its linear combination with the metric. Weyl tensor enters only quadratically and being identically zero on any conformally flat manifold does not contribute to conformally flat background solutions. Importantly, spatially flat FLRW metric is conformally flat.

(i) 它仅包含里奇标量和外尔张量，不包含里奇张量或其与度规的线性组合。外尔张量仅以二次项形式出现，且在任意共形平坦流形上恒为零，因此对共形平坦背景解没有贡献。重要的是，空间平坦的 FLRW 度规是共形平坦的。

(ii) as such, any solution already found in the literature with only $R\mathcal{F}_R(\Box)R$ piece in the action is a solution to equations of motion which one can derive from our new action.

(ii) 因此，文献中已找到的所有作用量仅含 $R\mathcal{F}_R(\Box)R$ 项的解，同样都是可从我们新作用量推导得到的运动方程的解。

(iii) linear perturbations of Weyl tensor are very simple using $(1+3)$ decomposition of the ADM formalism. These were computed in [41], and one can track computations relevant to our AID models in application to inflation to the end. Actually, perturbations of a possible term with any of the second- rank tensors (Ricci, Schouten, Einstein, or L -tensor) turn out to be very much complicated and seem to be intractable.

(iii) 利用 ADM 形式的 $(1+3)$ 分解，外尔张量的线性扰动形式非常简单，相关计算已在文献 [41] 中给出，我们可以完整梳理出适用于我们 AID 模型、应用到暴胀场景的计算。实际上，任何包含二阶张量 (里奇张量、斯豪滕张量、爱因斯坦张量或 L 张量) 的可能项，其扰动都会极为复杂，似乎难以处理。

The crux of this reduction is the observation that one can express, for example, $\Box L_{\mu\nu}$ as a combination of derivatives acting on Weyl tensor and R plus extra higher curvature terms. The following tensor identity was used:

约化的核心在于我们发现，可以将例如 $\Box L_{\mu\nu}$ 表示为作用于外尔张量和 R 的导数组合，再加上额外的高曲率项。推导用到了如下张量恒等式:

$$\begin{aligned}
&\frac{1}{D-3}\nabla^{\alpha}\nabla^{\beta}W_{\alpha\mu\nu\beta} \\
&= -\Box S_{\mu\upsilon} + \frac{1}{2(D-1)}\nabla_{\mu}\partial_{\upsilon}R + \frac{1}{D-2}W_{\rho\upsilon\mu\alpha}L^{\alpha\rho} \qquad (21) \\
&+\frac{D}{(D-2)^2}L_{\mu\alpha}L^{\alpha}_{\upsilon} - \frac{1}{(D-2)^2}g_{\mu\upsilon}L^2_{\alpha\beta} + \frac{1}{(D-1)(D-2)}RL_{\mu\upsilon}
\end{aligned}$$

where

其中

$$S_{\mu\nu} = \frac{1}{D-2}\left(R_{\mu\nu} - \frac{1}{2(D-1)}Rg_{\mu\nu}\right)$$

is the Schouten tensor. The higher curvature terms will either not contribute at all (at least on Minkowski background) or can be equivalently replaced by quadratic in curvature terms again. Still the action remains complicate without an obvious hope of solving equations of motion.

为斯豪滕张量。高曲率项要么完全没有贡献 (至少在闵氏背景下)，要么可以等价地重新表示为曲率的二次项。即便如此，作用量仍然十分复杂，看不出能求解运动方程的希望。

It is worth stressing that actions (18) and (19) are not fully equivalent. They are equivalent as long as at most linear perturbations around MSS are considered. As a consequence, non-MSS may be solutions to EOM derived from one action and not from another. For example, local R^2 inflationary background is a solution to EOM derived from action (18) and is not a solution as long as the quadratic term with a second-rank tensor is restored in the action. Furthermore, higher, i.e., 3-, 4-, ...,- point vertices and correlation functions are clearly different in these actions. It is however of course possible that the difference may not be important for particular models and under certain assumptions.

需要强调的是，作用量 (18) 和 (19) 并非完全等价，二者仅在讨论 MSS 附近的至多线性扰动时等价。因此，非 MSS 解可能是从一个作用量导出的运动方程的解，却不属于另一个。例如，局域 R^2 暴胀背景是从作用量 (18) 导出的运动方程的解，但只要作用量中恢复出含二阶张量的二次项，它就不再是解。此外，更高阶的点顶点 (即三点、四点……顶点) 和关联函数在这两个作用量中显然不同。当然，差异也有可能在特定模型和一定假设下并不重要。

Two more things to mention here is that first, we have not appealed to strings in deriving this, and, second, to this moment we have not said how many derivatives are needed or expected. As one can guess, shortly we will find that infinitely many.

这里还有两点需要说明: 第一，我们推导该结果时没有引入弦论相关内容；第二，到目前为止我们还没有说明需要或期望存在多少阶导数。不难猜到，我们很快就会发现，导数有无穷多阶。

Propagator

传播子

To see that infinitely many derivatives are inevitable, one has to write down propagators for the scalar ant tensor degrees of freedom coming out of this action. This can be done nowadays simply citing the original derivation in [21] and the outcome is

为了说明无穷多阶导数是不可避免的，我们需要写出该作用量给出的标量和张量自由度的传播子。如今我们只需引用文献 [21] 中的原始推导即可，结果如下

$$S_2 = \frac{1}{2}\int dx^4 \sqrt{-\bar{g}} h^{\perp\mu\nu}\left(\overline{\Box} - \frac{\bar{R}}{6}\right)\left\{\mathcal{P}\left(\overline{\Box}\right)\right\} h^{\perp}{}_{\mu\nu}, \tag{22}$$

with

其中

$$\mathcal{P}\left(\overline{\Box}\right) = 1 + \frac{2}{M_P^2}\lambda f_{R,0}\bar{R} + \frac{\lambda}{M_P^2}2\left(\overline{\Box} - \frac{\bar{R}}{3}\right)\mathcal{F}_W\left(\overline{\Box} + \frac{\bar{R}}{3}\right)$$

and

且

$$S_0 = -\frac{1}{2}\int dx^4 \sqrt{-\bar{g}}\phi\left(\overline{\Box} + \frac{\bar{R}}{3}\right)\{\mathcal{S}\left(\overline{\Box}\right)\}\phi, \tag{23}$$

with

满足

$$\mathcal{S}\left(\overline{\Box}\right) = 1 + \frac{2}{M_P^2}\lambda f_{R,0}\bar{R} - \frac{\lambda}{M_P^2}2\left(3\overline{\Box} + \bar{R}\right)\mathcal{F}_R\left(\overline{\Box}\right)$$

where $h^{\perp}$ is the transverse and traceless excitation of the metric containing graviton and ϕ is the scalaron.

式中 $h^{\perp}$ 是包含引力子的度规横截无迹激发，ϕ 是标量子。

Let us concentrate on the propagator for $h^{\perp}$. The logic of the higher derivative theories tells us that no more than one physical pole can be. That said, given that the massless graviton is already in the spectrum, thanks to the factor $\left(\overline{\Box} - \bar{R}/6\right)$ in (22), we must demand:

我们来重点讨论 $h^{\perp}$ 的传播子。高阶导数理论的逻辑告诉我们，最多只能存在一个物理极点。鉴于无质量引力子已经处于能谱中，这要归功于 (22) 式中的因子 $\left(\overline{\Box} - \bar{R}/6\right)$，因此我们必须要求:

$$\mathcal{P}\left(\overline{\Box}\right) = \exp\left(2\omega\left(\overline{\Box}\right)\right)$$

for some entire function $\omega\left(\overline{\Box}\right)$.

对某个整函数 $\omega\left(\overline{\Box}\right)$ 成立。

Then we see the following chain of arguments. We can take $\mathcal{P} = 1$ and this will be back GR. GR, however, is not great and was shown by Stelle that has a way improved form with a local R^2 term. The latter however requires W^2 to make the model renormalizable. And this in turn generates an extra pole and a ghost as it corresponds to have $\mathcal{F}_W = 1$. Therefore, we neither want $\mathcal{P} = 1$ nor we are happy with Stelle gravity.

接下来我们来看下述逻辑链: 若我们取 $\mathcal{P} = 1$ ，就回到了广义相对论 (GR)。但广义相对论并不完美，Stelle 证明，加入局域 R^2 项可以得到形式大幅改进的理论。不过后者需要 W^2 才能让模型可重整化，而这又会产生额外极点和鬼场，对应存在 $\mathcal{F}_W = 1$ 。因此我们既不想要 $\mathcal{P} = 1$ ，也不满意 Stelle 引力。

From here we come to a very simple purely algebraic observation that one needs an infinite number of derivatives in $\mathcal{F}$ -s because it is the only way to equate $\mathcal{P}$ and $\mathcal{S}$ to an exponent of an entire function. This advocates our passion to infinite derivative theories as it seems to be the only way to a well-defined gravity.

由此我们得到一个非常简单的纯代数结论: 在 $\mathcal{F}$ 中需要无穷多阶导数，因为这是将 $\mathcal{P}$ 和 $\mathcal{S}$ 化为整函数指数的唯一方法。这印证了我们对无穷导数引力理论的研究追求，它似乎是得到良定义引力的唯一途径。

However, one important issue arises here, exactly the one of ghosts of a kind described in the previous section. The above operators $\mathcal{P}$ and $\mathcal{S}$ can be made well defined for one and given value of R , say for $R = 0$, i.e., Minkowski. If, however, one wants to consider another background in the same action, new degrees of freedom will appear.

但这里出现了一个重要问题，就是上一节描述过的那类鬼场问题。对于给定的 R 取值，例如 $R = 0$ 即闵氏背景，上述算符 $\mathcal{P}$ 和 $\mathcal{S}$ 可以做到良定义。然而如果我们想在同一个作用量中考虑另一个背景，就会出现新的自由度。

One again can wonder: wait a second, what other backgrounds if you already said it all true around a given MSS. Isn't it fixed by that? Our answer again is "no." We indeed have done all the derivation above in an attempt to see a most general action to study linear perturbations around MSS. This is true. It is however not obvious from any principle that the above action (18) is not enough to describe gravity in our world. As such nothing stops us from exploring this action deeper to see whether it can satisfy our other needs. In particular it can accommodate inflationary Starobinsky solution as will be seen in the next section. This solution has both nearly de Sitter and nearly Minkowski stages. A problem with infinite derivatives is that only one of those can be made really ghost-free.

有人可能会疑惑: 等等，你已经说了在给定最大对称背景 (MSS) 下所有结论都成立，那怎么还会有其他背景？难道结论不是固定的吗？我们的回答仍然是“不”。我们确实是在最大对称背景的线性扰动下推导得到了上述最一般的作用量，这一点没错。但并没有任何原理明确说明上述作用量 (18) 不足以描述我们世界的引力。因此没有什么能阻碍我们更深入地研究这个作用量，看看它能否满足我们的其他需求。下一讲我们会看到，它尤其可以容纳 Starobinsky 暴胀解。该解同时具有近德西特阶段和近闵氏阶段。无穷导数的问题在于，两个阶段中只有一个能做到真正无鬼。

On the other hand, one indeed can say that just a quadratic in curvature action is not enough. One of the plausible generalizations is to extend action (19) as follows:

另一方面，人们确实可以提出，仅包含曲率二次项的作用量是不够的。一种合理的推广是将作用量 (19) 拓展如下:

$$S=\int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R+\frac{\lambda}{2}\left(R\mathcal{F}_R\left(\Box\right)R\right.\right.$$

$$\left.\left.+\left(\lambda_0+\frac{R}{\mathcal{M}_s^2}\right)W_{\mu\nu\rho\sigma}\mathcal{F}_W\left(\Box\right)W^{\mu\nu\rho\sigma}\right)-\Lambda\right]. \tag{24}$$

The new cubic curvature term will not influence propagator of the graviton around Minkowski but can compensate the appearance of the R factors in (22) in $\mathcal{P}$ and as such stabilize the system. This scenario is elaborated in [42]. However, this cure is not enough as this does not void the problem of extra degrees of freedom in the scalar sector. Of course a similar trick in principle can be implemented, but then a lot of beautiful solutions will be lost. In particular, Starobinsky solution will not be a solution anymore (or at least keeping it as a solution will be extraordinary difficult). This will be explicitly clear from the solution construction we are coming to in the next section.

新的曲率三次项不会影响闵氏背景下引力子的传播子，但可以抵消 $\mathcal{P}$ 中 (22) 式 R 因子的出现，从而稳定系统。该方案已在文献 [42] 中详细研究。但这个解决方法并不足够，因为它没有消除标量 sector 中额外自由度的问题。当然原则上也可以用类似的技巧处理，但会丢失很多优美的解。尤其是 Starobinsky 解将不再是该理论的解 (或者说，至少要保留它为解会极其困难)，这一点我们在下一节构造解的时候会看得很清楚。

To conclude this section, we would like to say that a possibility of containing ghosts on different backgrounds by choosing a shape of form factors and as such guaranteeing nongrowing trajectories of new degrees of freedom is a possibility and its exploration is under construction in ongoing projects.

最后在本节结尾，我们想要说明: 通过选择形状因子，确实有可能在不同背景下控制鬼场，从而保证新自由度的轨迹不增长，这一方向目前正在多个在研项目中探索。

Classical Dynamics of AID Quadratic Gravity Without Λ

不含 Λ 的 AID 二次引力经典动力学

As said before, we may hope to find not only pure MSS backgrounds as a solution to equations of motion. This section will illuminate how we can indeed find some nontrivial backgrounds.

如前所述，我们有望不止找到纯 MSS 背景作为运动方程的解，本节将阐明我们如何切实得到一些非平庸背景。

EOM and Solution Construction

运动方程与解构造

Let's start with the most simple setup without a cosmological constant term. To proceed with the actual computation, we recite action (19) dropping the cosmological term Λ :

我们从不存在宇宙学常数项的最简 setup 开始讨论。为开展实际计算，我们重新写出作用量 (19)，去掉宇宙学项 Λ :

$$S=\int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R+\frac{\lambda}{2}\left(R\mathcal{F}_R\left(\Box\right)R+W_{\mu\nu\rho\sigma}\mathcal{F}_W\left(\Box\right)W^{\mu\nu\rho\sigma}\right)\right]. \tag{25}$$

This action was studied and many technical details were elaborated in [41, 43, 44] . We are going to use them without extensive further referencing.

该作用量已在 [41, 43, 44] 中被研究，许多技术细节也已在其中展开，我们将直接使用这些结论，不再额外大量引用。

EOM which one can derive from action (25) (see [45]) reads:

可从作用量 (25) 推导出的运动方程 (参见 [45]) 如下:

$$E^\mu_\upsilon\equiv-\left(M_P^2+2\lambda\mathcal{F}\left(\Box\right)R\right)G^\mu_\upsilon-\frac{1}{2}\lambda\delta^\mu_\upsilon R\mathcal{F}\left(\Box\right)R+2\lambda\left(\nabla^\mu\partial_\upsilon-\delta^\mu_\upsilon\Box\right)\mathcal{F}\left(\Box\right)R$$

$$+\lambda\mathcal{L}^\mu_\upsilon-\frac{\lambda}{2}\delta^\mu_\upsilon\left(\mathcal{L}^\sigma_\sigma+\bar{\mathcal{L}}\right)+2\lambda\left(R_{\alpha\beta}+2\nabla_\alpha\nabla_\beta\right)\mathcal{F}_W\left(\Box\right)W^{\alpha\beta\mu}_\upsilon+\mathcal{O}\left(W^2\right)=0,$$

(26)

$$\mathcal{L}^\mu_\upsilon=\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}\partial^\mu R^{(l)}\partial_\upsilon R^{(n-l-1)},\ \bar{\mathcal{L}}=\sum_{n=1}^{\infty}f_n\sum_{l=0}^{n-1}R^{(l)}R^{(n-l)},\ R^{(l)}\equiv\Box^l R.$$

The trace equation reads:

迹方程如下:

$$E=\left(M_P^2-6\lambda\Box\mathcal{F}_R\left(\Box\right)\right)R-\lambda\left(\mathcal{L}^\mu_\mu+2\bar{\mathcal{L}}\right)+\mathcal{O}\left(W^2\right)=0. \tag{27}$$

Terms linear in Weyl tensor are not present in the trace equation because their trace vanishes by construction on any space-time. Terms $\mathcal{O}\left(W^2\right)$ can be found in [46].

Weyl 张量的线性项未出现在迹方程中，因为这类项的迹按构造在任意时空上都为零。$\mathcal{O}\left(W^2\right)$ 项可在 [46] 中找到。

As a simplest scenario, we are interested in cosmological solutions of the spatially flat FLRW type. First this implies that the Weyl tensor vanishes, and as such it does not manifest itself in the trace equation neither in the background nor in linear perturbations. Second, such solutions for the metric are space-homogeneous and isotropic. This means that system of Eq. (26) has essentially two distinct equations. The standard choice is the trace equation and the $\begin{pmatrix}0\\0\end{pmatrix}$-equation. However, presence of Bianchi identities guarantees that given we have a solution to the trace equation with zero RHS, then it will be a solution to the whole system of equations modulo a possible radiation source (which is conserved and is traceless). We are thus focused on solving the trace equation (27) which is a nonlinear differential (non-local) equation on R , and all the differential operators are of the form of d’ Alembertian.

作为最简单的情形，我们关注空间平坦 FLRW 型的宇宙学解。首先，这意味着 Weyl 张量为零，因此无论在背景还是线性微扰中，Weyl 张量都不会出现在迹方程中。其次，这类度规解是空间均匀且各向同性的，这说明式 (26) 方程组本质上包含两个独立方程。标准选择是迹方程和 $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 方程。但 Bianchi 恒等式保证，若我们得到了右侧为零的迹方程的解，那么除去可能的辐射源 (辐射源守恒且无迹)，该解就是整个方程组的解。因此我们专注于求解迹方程 (27)，这是一个关于 R 的非线性微分 (非局部) 方程，所有微分算符都满足达朗贝尔算符形式。

We start solving the trace equation by reminding that originally it was proposed in [47] to use an ansatz:

我们开始求解迹方程，首先回顾文献 [47] 最初提出使用如下拟设:

$$\Box R = r_1 R \tag{28}$$

to construct solutions. First we note that the original ansatz also had a free constant term r_2 in the right-hand side, but it is not compatible with the absence of the cosmological term. Also we note that this ansatz was indeed helpful to construct several exact solutions to equation of motion.

来构造解。首先我们注意，原始拟设的右侧原本还包含一个自由常数项 r_2，但它与不存在宇宙学项的前提不兼容。我们还要指出，这个拟设确实有助于构造多个运动方程的精确解。

One of the most important implications of using the above ansatz is that Starobinsky inflation [48] can be embedded in the infinite derivative gravity. This is because the above ansatz is the trace equation of a local R^2 gravity.

使用上述拟设最重要的结论之一，就是 Starobinsky 暴胀 [48] 可以被嵌入无限导数引力中。这是因为上述拟设就是局部 R^2 引力的迹方程。

It is instructive to show sketchy how the technique works. Substituting (28) into (27), we restore the result obtained in [47]:

简要展示该方法的工作原理是有启发性的。将 (28) 代入 (27)，我们复现了 [47] 中得到的结果:

$$\left(M_P^2 - 6\lambda r_1 \mathcal{F}_R(r_1)\right) R - \lambda \mathcal{F}_R^{(1)}(r_1)\left(\partial^\mu R \partial_\mu R + 2 r_1 R^2\right) = 0. \tag{29}$$

The way to solve the latter equation is to assume the algebraic conditions:

求解后一方程的方法是假设如下代数条件:

$$\mathcal{F}_R^{(1)}(r_1) = 0, \frac{M_P^2}{2\lambda} = 3 r_1 \mathcal{F}_1, \text{ where } \mathcal{F}_1 \equiv \mathcal{F}_R(r_1). \tag{30}$$

Since we constrain here only parameters, we get the most what we can using (28). If we do not impose the above conditions, then we must satisfy additional equation on R which is the present equation (29). This can be shown to trivialize possible solutions to just one $R = 0$. Below is a subsection addressing this approach.

由于我们仅对参数做约束，使用拟设 (28) 已经得到了我们能得到的全部结果。如果不施加上述条件，那么我们必须满足当前方程 (29) 给出的关于 R 的附加方程，可以证明这会将所有可能的解退化为唯一的 $R=0$ 。下文小节将讨论这一思路。

Solving (29) Without Using (30)

不使用 (30) 求解 (29)

Let us find all solutions of Eq. (29):

我们来求方程 (29) 的所有解:

$$AR = B\left(\dot{R}^2 + 2r_1R^2\right),\ A = M_P^2 - 6\lambda r_1\mathcal{F}_R\left(r_1\right),\ B = \lambda\mathcal{F}_R^{(1)}\left(r_1\right) \tag{31}$$

under the condition that R satisfies the Eq. (28)

在 R 满足方程 (28) 的条件下

$$\Box R = r_1R \tag{32}$$

with $r_1 > 0$ and a spatially flat FLRW space-time is assumed. The simplest way to solve this problem is to reduce the differential equation (31), which is the second order with respect to the Hubble function $H \equiv \dot{a}/a$ (the scale factor $a\left(t\right)$ itself does not enter due to invariance under rescaling of all spatial coordinates), to an expression containing $R\left(t\right)$ only which should be an identity if R is not a constant.

这里假定了 $r_1 > 0$ 和空间平坦的 FLRW 时空。求解该问题的最简方法是将关于哈勃函数 $H \equiv \dot{a}/a$ 的二阶微分方程 (31)(由于该方程在所有空间坐标的尺度变换下不变，标度因子 $a\left(t\right)$ 本身不进入方程) 约化为仅含 $R\left(t\right)$ 的表达式，若 R 不是常数，该表达式应为恒等式。

Note first that one such solution is $R \equiv 0$, and then no conditions on A, B and r_1 arise.

首先注意，$R \equiv 0$ 就是一个这样的解，此时对 A, B 和 r_1 没有任何约束。

Let us assume further that R is not identically equal to 0 . Let $B \neq 0$. Then

下面我们假设 R 不恒等于 0，令 $B \neq 0$ ，则有

$$\dot{R} = -\sqrt{R\left(\frac{A}{B} - 2r_1R\right)}, \tag{33}$$

where we take the minus sign for $\dot{R}$ corresponding to decrease of R with time for definiteness. If $r_1 \neq 0$ and $R \neq 0, \dot{R} \neq 0$ too. By differentiating Eq. (31) with respect to time and dividing both its sides by $\dot{R}$, one gets:

为明确起见，我们对 $\dot{R}$ 取负号，对应 R 随时间减小。如果 $r_1 \neq 0$ 且 $R \neq 0, \dot{R} \neq 0$ 也满足该条件。将方程 (31) 对时间求导，再将方程两边同时除以 $\dot{R}$ ，可得:

$$\frac{A}{2B} = \ddot{R} + 2r_1 R = -3H\dot{R} + r_1 R. \tag{34}$$

From Eqs. (33) and (34), the expression for $H(t)$ follows:

由方程 (33) 和 (34)，可得 $H(t)$ 的表达式为:

$$H = \frac{1}{6}\sqrt{\frac{A}{BR} - 2r_1}. \tag{35}$$

Differentiating Eq. (35) and using Eq. (33) once more, we arrive to

对方程 (35) 求导并再次利用方程 (33)，我们得到

$$\dot{H} = \frac{A}{12BR}. \tag{36}$$

From this the expression

由此可得表达式

$$R \equiv 6\dot{H} + 12H^2 = \frac{5A}{6BR} - \frac{2r_1}{3} \tag{37}$$

follows that cannot be satisfied for $\dot{R} \neq 0$. Thus, the only remaining possibility is $B = 0$ and then $A = 0$ if $R \neq 0$.

表明该式对 $\dot{R} \neq 0$ 无法成立。因此，仅剩下的可能是 $B = 0$ ，此时若 $R \neq 0$ 成立则可得 $A = 0$ 。

So, all solutions of Eq. (31) under the condition (32) are given either by $R \equiv 0$ or by $A = B = 0$. However, since $R \equiv 0$ is a particular solution of Eq. (32) (or (28)), too, we notice that this case having zero measure in the space of initial conditions for Eq. (27) does not require special consideration further.

因此，条件 (32) 下方程 (31) 的所有解可表示为 $R \equiv 0$ 或 $A = B = 0$ 。但由于 $R \equiv 0$ 本身也是方程 (32)(即方程 (28)) 的一个特解，我们注意到，该情况在方程 (27) 初始条件空间中的测度为零，后续无需单独讨论。

We finally note that one can extend the above proof to any metric which is space-homogeneous in the synchronous frame.

最后我们指出，上述证明可以推广至同步坐标系中任意空间均匀的度规。

Thus, if an ansatz (28) is imposed, then we are forced to use conditions (30) but what if we go beyond an ansatz?

因此，若我们采用假设 (28)，就必须使用条件 (30)，但如果我们不局限于该假设呢?

Proof That (28) Is a General Solution to (27)

证明 (28) 是 (27) 的通解

Now we formulate the main claim of this section and in fact a very important statement for the development of AID quadratic gravity theories in general.

现在我们阐述本节的核心命题，实际上这也是对整体 AID 二次引力理论发展而言非常重要的一个结论。

Proposition. Equation (28) in combination with conditions (30) provides the most general solution to the trace equation (27) if:

命题: 若满足以下条件，方程 (28) 结合条件 (30) 即为迹方程 (27) 的最通解:

(i) the metric is of a spatially flat FLRW type and

(i) 度规为空间平坦 FLRW 型，且

(ii) the Fourier harmonics form a basis on the domain of functions of interest on the space-time manifold.

(ii) 傅里叶谐波构成时空流形上目标函数域的一组基。

Let us start with noting that having a physical attitude to the problem, we formulate here sufficient and not obligatory necessary conditions.

首先我们说明，从物理问题的角度出发，我们此处给出的是充分条件而非必要条件。

The first condition (i) just serves for the setting of the present paper to discuss a space-homogeneous inflationary space-time and is simple to account. Technically during the proof, the only property to be exploited will be the space homogeneity of the metric in the synchronous frame. As such the proof itself can be applied to more general space-homogeneous metrics, for example, to anisotropic backgrounds like Bianchi I or other. However, we need the metric to be conformally flat to eliminate the Weyl tensor-squared terms in the trace equation (27). This would allow us to claim that the restrictions imposed by this proposition on the space of solutions apply to the full trace equation. Only for this purpose we stick to spatially flat FLRW metrics only. This implies that given the Weyl tensor-dependent term is not included in the action (25), one can relax condition (i) to: (i) the metric is space-homogeneous in the synchronous frame.

第一个条件 (i) 只是适配本文研究空间均匀膨胀时空的设定，很容易满足。从技术层面看，证明过程中仅需要用到同步坐标系中度规的空间均匀性这一性质，因此该证明本身也可推广至更一般的空间均匀度规，例如比安基 I 型这类各向异性背景。但我们需要度规具有共形平坦性，以消去迹方程 (27) 中的外尔张量平方项，这样才能保证本命题对解空间施加的约束适用于完整的迹方程。我们仅采用空间平坦 FLRW 度规完全是出于这一目的。这意味着，如果作用量 (25) 本身不包含依赖外尔张量的项，那么条件 (i) 可以放宽为:(i) 度规在同步坐标系中是空间均匀的。

The second condition (ii) needs more explanations though. We name Fourier harmonics the eigenfunctions of the d'Alembertian operator such that

不过第二个条件 (ii) 需要更多解释。我们将达朗贝尔算符的本征函数称为傅里叶谐波，满足

$$\Box\varphi_i = w_i\varphi_i, \tag{38}$$

where w_i are constants. Generically we expect the spectrum of the d' Alembertian is continuous even though this is not crucial. We name φ_i the Fourier harmonics in analogy with the flat space-time where they reduce to the plane waves which in turn are used to define the Fourier transform. A crucial property of the Fourier transform in the flat space-time is that the corresponding harmonics form a basis in the domain of square-integrable functions L_2 . Or in other words, any square-integrable function can be presented as a linear superposition of plane waves. In our model the situation seems to be more involved as a priori these nice properties of the Fourier transform in the flat space-time cannot be elevated to a curved background.

其中 w_i 为常数。一般而言，我们认为达朗贝尔算符的谱是连续的，尽管这一点并非关键。我们将 φ_i 命名为傅里叶调和函数，这是类比平直时空的情况——在平直时空中它们退化为平面波，而平面波正是用来定义傅里叶变换的。平直时空傅里叶变换的一个关键性质是，对应的调和函数在平方可积函数域 L_2 中构成一组基。换句话说，任意平方可积函数都可以表示为平面波的线性叠加。在我们的模型中，情况似乎更为复杂，因为平直时空傅里叶变换的这些优良性质先天无法直接推广到弯曲背景下。

It is known from the spectral analysis of the Beltrami-Laplace (BI) operator on Riemannian manifolds that indeed the eigenmodes of the BI operator form a basis in L_2 as long as the manifold is compact or has a boundary [49]. In most cosmological applications, the space-time manifold is however pseudo-Riemannian (i.e., the metric is not positively defined and d'Alembertian operator replaces the BI operator), non-compact, and without a boundary. In this situation general theorems do not help and presently one has to consider systems case-by-case. Paper [50] provides an explicit proof that in two notable cases of dS and (A)dS space-times, indeed the eigenmodes of the d'Alembertian operator form a basis for square-integrable functions. This remains valid in a special situation when there is no spatial dependence present. It is an important situation though since in the vast majority cosmologically viable backgrounds are space-homogeneous. Naturally, it is the case for the present paper as well. Technically, this implies that the d'Alembertian operator lacks of spatial derivatives and eigenmodes φ_i depend on time only.

从黎曼流形上贝尔特拉米-拉普拉斯 (BI) 算符的谱分析可知，只要流形是紧致的或带有边界，BI 算符的本征模确实能在 L_2 中构成一组基 [49]。但大多数宇宙学应用中，时空流形是伪黎曼的 (即度规不是正定的，此时用达朗贝尔算符代替 BI 算符)、非紧致且无边界，这种情况下通用定理并不适用，目前只能逐例分析。文献 [50] 给出了明确证明: 在德西特空间 (dS) 和 (反) 德西特空间 ((A)dS) 这两个重要情形中，达朗贝尔算符的本征模确实能为平方可积函数构成一组基，该结论在不存在空间依赖的特殊情形下仍然成立。这种情形十分重要，因为绝大多数宇宙学可行背景都是空间均匀的，本文的研究也自然属于这种情形。从技术层面看，这意味着达朗贝尔算符不含空间导数，本征模 φ_i 仅依赖于时间。

Coming to physical grounds, we stress that the regime of the space-time evolution of interest in the present paper is the nearly dS expansion. This in combination with the results in [50] provides some hint that our condition (ii) in the proposition above is sensible. However, there is one more physically important argument why a physically viable space-time must have such a structure that the d'Alembertian operator eigenmodes form a basis. Namely, we expect that our model can be quantized. To have this happen, we have to have a vacuum and creation and annihilation operators which in the canonical quantization scheme appear as operator coefficient in front of Fourier modes in which a given classical solution is decomposed. Given a situation that Fourier modes do not form a basis (i.e., the set of modes is not enough to represent any function), we will hit a problem that certain classical configurations cannot be quantized in a canonical way. This simple consideration shows that the fact that eigenmodes of the d'Alembertian operator form a basis is necessary to implement the canonical quantization scheme. This gives us even a stronger hint that we indeed want the condition (ii) in the proposition to be satisfied.

从物理角度出发，我们强调本文研究的时空演化区域是近似德西特膨胀。结合文献 [50] 的结果，这暗示上述命题中的条件 (ii) 是合理的。不过, 还有另一个更重要的物理论据说明, 物理可行的时空必须具有达朗贝尔算符本征模构成一组基的结构: 我们期望该模型是可量子化的。要实现量子化，我们需要存在真空以及产生湮灭算符，在正则量子化方案中，这些算符作为分解经典解的傅里叶模的算符系数出现。如果傅里叶模不能构成一组基 (即模的集合不足以表示任意函数)，我们就会遇到部分经典构型无法按正则方式量子化的问题。这一简单推导表明，达朗贝尔算符的本征模构成一组基是实现正则量子化方案的必要条件。这进一步说明，我们确实需要满足命题中的条件 (ii)。

Finally, we do not specify explicitly the domain of functions on which the completeness of the Fourier decomposition is true. We presume that in most cases, we need to have it either for functions from L_2 or functions with a compact support which is a more plausible case as long as time evolution of a classical system is considered. This will be noted just below as well.

最后，我们没有明确指定傅里叶分解完备性成立的函数定义域。我们推测，大多数情况下，完备性要么对来自 L_2 的函数成立，要么对具有紧支集的函数成立；在研究经典系统时间演化时，紧支集的情况更合理。这一点我们也会在下文说明。

Therefore, in proving the proposition, we assume that the scalar curvature R as any function can be represented as

因此，在证明该命题时，我们假设标量曲率 R 和任意函数都可以表示为

$$R = \sum_i \varphi_i, \ \Box\varphi_i = w_i\varphi_i \tag{39}$$

and w_i are constants. Possible constants in front of φ_i in the decomposition of R are absorbed inside of φ_i for simplicity. Few comments are in order here. First, one should not be confused with the fact that R itself depends on the metric as for the time being it is just some function of time. Second, one should not worry about possible nontrivial asymptotics of R in past or future infinities (which may render it non-square integrable) and consider only a given time interval during which our model describes the evolution of the Universe. This will waive doubts of the square integrability since the function gains the compact support by construction. In other words it is equivalent to saying that we work in a given coordinate patch. Also, here we explicitly come to the special situation mentioned above that all functions depend on time only since a space-homogeneous background is considered. The corresponding simplification will become crucial to fulfill the proof.

且 w_i 为常数。为简化起见，R 分解中 φ_i 前可能存在的常数已吸收进 φ_i。此处做几点说明: 首先，不要混淆 R 本身依赖于度规这一点，因为目前它只是一个关于时间的函数。其次，不必担心 R 在过去或未来无穷远处可能存在非平凡渐近 (这可能导致它不是平方可积的)，我们只需要考虑模型描述宇宙演化的给定时间区间即可。这样就消除了平方可积性的疑问，因为根据构造，函数自动获得紧支集。换句话说，这等价于我们在给定坐标卡内工作。此外，由于我们研究的是空间均匀背景，此处明确对应上文提到的特殊情况: 所有函数仅依赖时间。该简化对完成证明至关重要。

Using (39) one readily computes:

利用 (39) 可以很容易算出:

$$\square^l R = \sum_i w_i^l \varphi_i, \ \mathcal{F}_R(\square) R = \sum_i \mathcal{F}_R(w_i)\varphi_i \tag{40}$$

and further

进一步可得

$$\mathcal{L}^\mu_\mu = \sum_{i,j} \omega_{ij}\partial^\mu\varphi_i\partial_\mu\varphi_j, \ \overline{\mathcal{L}} = \sum_{i,j}\omega_{ij}w_j\varphi_i\varphi_j, \ \omega_{ij} = \frac{\mathcal{F}_R(w_i) - \mathcal{F}_R(w_j)}{w_i - w_j}.$$

(41)

Notice that for $i = j$, we have to use the Taylor series expansion to obtain $\omega_{ii} = \mathcal{F}_R^{(1)}(w_i)$ where the superscript $^{(1)}$ denotes the derivative with respect to an argument. Substituting all of that into (27) and accounting that the Weyl tensor vanishes, one yields:

注意，对于 $i = j$，我们需要使用泰勒级数展开得到 $\omega_{ii} = \mathcal{F}_R^{(1)}(w_i)$，其中上标 $^{(1)}$ 表示对宗量求导。将所有结果代入 (27)，并考虑到外尔张量为零，我们得到:

$$M_P^2\sum_k \varphi_k - 6\lambda\sum_k w_k\mathcal{F}_R(w_k)\varphi_k - \lambda\sum_{i,j}\omega_{ij}\left(\partial^\mu\varphi_i\partial_\mu\varphi_j + (w_i + w_j)\varphi_i\varphi_j\right) = 0.$$

(42)

To prove the proposition, we have to show that no (nontrivial) solutions to (42) exist as long as R is a superposition of more than a single Fourier eigenmode.

为证明该命题，我们需要说明: 只要 R 是多个傅里叶本征模的叠加，(42) 就不存在 (非平凡) 解。

First we note that the technique of equating coefficient to zero does not work in this general case. Indeed, the quadratic in φ_i term in (42) can be eliminated by requiring $\mathcal{F}_R(w_i) = \mathcal{F}_R(w_j)$ and $\mathcal{F}_R^{(1)}(w_i) = 0$ for any i, j. This being substituted into the terms linear in φ_i yields:

首先我们指出，令系数为零的方法在这个一般情况下不适用。事实上，对任意 i, j，我们可以通过要求 $\mathcal{F}_R(w_i) = \mathcal{F}_R(w_j)$ 和 $\mathcal{F}_R^{(1)}(w_i) = 0$ 消去 (42) 中 φ_i 的二次项。将这个要求代入 φ_i 的一次项后可得:

$$M_P^2 \sum_k \varphi_k - 6\lambda \mathcal{F}_R(w_1) \sum_k w_k \varphi_k = 0.$$

Since however different φ_k are eigenfunctions of d'Alembertian with different eigenvalues, they are linearly independent. This means that in order to satisfy the latter equation, we must require $M_P^2 - 6\lambda \mathcal{F}_R(w_1) w_k = 0$ for each k and as such, all w_k are equal. We thus effectively come back to the situation $R = \varphi_1$ like it is served by (28).

但由于不同的 φ_k 是达朗贝尔算符对应不同本征值的本征函数，它们线性无关。这意味着为了满足上述方程，我们必须对每个 k 要求 $M_P^2 - 6\lambda \mathcal{F}_R(w_1) w_k = 0$ ，因此所有 w_k 都相等。这样我们实际上回到了 (28) 对应的情况 $R = \varphi_1$ 。

Thus, we must keep the quadratic terms in (42) and solve it as a differential equation on φ_i . Satisfying (42) will necessarily produce stringent constraints since the resulting solution for R must be identical to the Ricci scalar constructed from the metric. Note that in the beginning of the proof, we have mentioned that R is just some function of time. Here we explicitly make reference to its relation to the metric. This, however, in no way complicates the use of desired spectral properties of the d'Alembertian.

因此，我们必须保留 (42) 中的二次项，将其作为 φ_i 上的微分方程求解。满足 (42) 必然会带来严格约束，因为得到的 R 解必须与由度量构造的里奇标量完全一致。请注意，在证明开篇我们已经提到，R 只是时间的某个函数，此处我们明确给出它与度量的关联。但这完全不会影响达朗贝尔算符预期谱性质的使用。

Going further one can pass to modified quantities $\widetilde{\varphi}_i = \varphi_i + c_i$ where we have done shifts by constants defined as

进一步地，我们可以变换为修正量 $\widetilde{\varphi}_i = \varphi_i + c_i$ ，我们已经通过如下定义的常数完成了平移:

$$2\lambda \sum_j \omega_{kj} (w_k + w_j) c_j + M_P^2 - 6\lambda w_k \mathcal{F}_R(w_k) = 0 \text{ for each } k. \tag{43}$$

Hence we rewrite (42) as

因此我们将 (42) 重写为:

$$\sum_{i,j} \omega_{ij} \dot{\widetilde{\varphi}}_i \dot{\widetilde{\varphi}}_j - \sum_{i,j} \omega_{ij} (w_i + w_j) \widetilde{\varphi}_i \widetilde{\varphi}_j = c = -\sum_{i,j} \omega_{ij} (w_i + w_j) c_i c_j, \tag{44}$$

where we have used the fact that all φ_i are space-homogeneous and depend only on time. The sign change in front of $\sim \dot{\widetilde{\varphi}}_i^2$ is due to the signature of the metric, and also we assume that $g_{00} = -1$. Also a common factor λ has been cancelled. Interestingly, we recognize in the latter formula the conserved integral of energy originating from a sigma-model-type dynamical system.

其中我们利用了所有 φ_i 都是空间齐次、仅依赖时间这一性质。$\sim \dot{\widetilde{\varphi}}_i^2$ 前的符号变化由度量的符号差导致，此外我们假设 $g_{00} = -1$ ，并且已经消去了公因子 λ 。有意思的是，我们可以在最后这个公式中认出，它源于 sigma 模型型动力系统的守恒能量积分。

To make the succeeding analysis more transparent, we rewrite the last formula using matrix notations as follows:

为了让后续分析更清晰，我们使用矩阵记号将上式重写如下:

$$\widetilde{\mathbf{R}}^{\mathrm{T}}\omega\widetilde{\mathbf{R}}-\widetilde{\mathbf{R}}^{\mathrm{T}}\left(\mathbf{w}\omega+\omega\mathbf{w}\right)\widetilde{\mathbf{R}}=c, \tag{45}$$

where $\widetilde{\mathbf{R}}$ is a vector made of $\widetilde{\varphi}_i$ and $\mathbf{w} = \operatorname{diag}\left(w_1, w_2, \ldots\right)$ and $\boldsymbol{\omega}$ is a matrix formed by ω_{ij} . We use a simple transposition as all quantities are real and matrices are symmetric from the physical origin of the problem. We diagonalize matrix $\boldsymbol{\omega}$ by choosing an appropriate matrix $\mathbf{D}$. We can always do this because if $\boldsymbol{\omega}$ cannot be diagonalized, then some values w_i are identical and we must just drop equivalent terms from decomposition (39). Denoting $\mathbf{d}^2 = \mathbf{D}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{D}$ and using further redefined functions $\mathbf{Q} = \mathbf{d}\mathbf{D}^{\mathrm{T}}\widetilde{\mathbf{R}}$, we get a canonically normalized diagonal term with derivatives. The whole expression transforms as

其中 $\widetilde{\mathbf{R}}$ 是由 $\widetilde{\varphi}_i$ 和 $\mathbf{w} = \operatorname{diag}\left(w_1, w_2, \ldots\right)$ 构成的向量，$\boldsymbol{\omega}$ 是由 ω_{ij} 构成的矩阵。由于所有量都是实量，且该问题的物理起源决定了矩阵是对称的，因此我们只需使用简单转置。我们通过选取合适的矩阵 $\mathbf{D}$ 将矩阵 $\boldsymbol{\omega}$ 对角化。我们总能做到这一点: 若 $\boldsymbol{\omega}$ 无法对角化，说明存在若干相同的 w_i 值，我们只需从分解式 (39) 中删去等价项即可。记 $\mathbf{d}^2 = \mathbf{D}^{\mathrm{T}}\boldsymbol{\omega}\mathbf{D}$ 并进一步重新定义函数 $\mathbf{Q} = \mathbf{d}\mathbf{D}^{\mathrm{T}}\widetilde{\mathbf{R}}$ 后，我们得到了带导数的规范归一化对角项。整个表达式变换为:

$$\dot{\mathbf{Q}}^{\mathrm{T}}\dot{\mathbf{Q}}-\mathbf{Q}^{\mathrm{T}}\mathbf{v}\mathbf{Q}=c, \tag{46}$$

where $\mathbf{v} = \mathbf{d}^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{w}\mathbf{D}\mathbf{d}+\mathbf{d}\mathbf{D}^{\mathrm{T}}\mathbf{w}\mathbf{D}\mathbf{d}^{-1}$. We can simplify the things even more by diagonalizing the matrix $\mathbf{v}$ choosing an appropriate matrix $\mathbf{M}$. Denoting $\mathbf{m}^2 = \mathbf{M}^{\mathrm{T}}\mathbf{v}\mathbf{M}$ and redefining $\mathbf{P} = \mathbf{M}^{\mathrm{T}}\mathbf{Q}$, we get:

其中 $\mathbf{v} = \mathbf{d}^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{w}\mathbf{D}\mathbf{d}+\mathbf{d}\mathbf{D}^{\mathrm{T}}\mathbf{w}\mathbf{D}\mathbf{d}^{-1}$ 。我们可以通过选取合适的矩阵 $\mathbf{M}$ 将矩阵 $\mathbf{v}$ 对角化，进一步简化问题。记 $\mathbf{m}^2 = \mathbf{M}^{\mathrm{T}}\mathbf{v}\mathbf{M}$ 并重新定义 $\mathbf{P} = \mathbf{M}^{\mathrm{T}}\mathbf{Q}$ 后，我们得到:

$$\dot{\mathbf{P}}^{\mathrm{T}}\dot{\mathbf{P}}-\mathbf{P}^{\mathrm{T}}\mathbf{m}^2\mathbf{P}=c. \tag{47}$$

Here the most crucial achievement that matrix $\mathbf{m}$ is diagonal.

此处最关键的结果是矩阵 $\mathbf{m}$ 为对角矩阵。

Differentiating the latter equation with respect to the time t , one gets:

对上述方程关于时间 t 求导，可得:

$$\dot{\mathbf{P}}^{\mathrm{T}}\left(\ddot{\mathbf{P}}-\mathbf{m}^2\mathbf{P}\right)=0. \tag{48}$$

As noticed above, all φ_i are linearly independent and the same are P_i . Indeed, since the matrices which define the quadratic form are nondegenerate, this guarantees that P_i are linearly independent. We thus can consider only the second-order linear equations in the latter expression as all of them must be satisfied independently. Since moreover matrix $\mathbf{m}$ is diagonal, we readily find each P_i as

如上所述，所有 φ_i 线性无关，P_i 也同样线性无关。事实上，由于定义二次型的矩阵是非退化的，这就保证了 P_i 线性无关。因此我们只需考虑上述表达式中的二阶线性方程，因为所有方程都必须独立成立。此外由于矩阵 $\mathbf{m}$ 是对角矩阵，我们可以直接得到每个 P_i 的表达式:

$$P_i = P_{i+}e^{m_i t} + P_{i-}e^{-m_i t}, \tag{49}$$

where we have assumed $\mathbf{m} = \operatorname{diag}(m_1, m_2, \ldots)$.

其中我们假设了 $\mathbf{m} = \operatorname{diag}(m_1, m_2, \ldots)$。

Returning to (39) we can rewrite the corresponding expression in matrix notations as well. That is,

回到 (39)，我们也可以用矩阵记号重写对应的表达式，即:

$$\ddot{\mathbf{R}} + 3H\dot{\mathbf{R}} + \mathbf{w}\mathbf{R} = 0. \tag{50}$$

Note that the latter equation is valid for any space-homogeneous metric as long as $g_{00} = -1$ and as such H is the Hubble function only in the case of a spatially flat FLRW metric. Passing to variables P_i , we get:

请注意，后一个方程对任何空间齐次度量都成立，只要满足 $g_{00} = -1$ ，因此仅当空间平坦 FLRW 度量时，H 才是哈勃函数。转换为变量 P_i 后，我们得到:

$$\ddot{\mathbf{P}} + 3H\dot{\mathbf{P}} + \boldsymbol{\mu}\mathbf{P} = \chi\mathbf{c}, \tag{51}$$

where $\boldsymbol{\mu} = \mathbf{M}^{\mathrm{T}}\mathbf{d}\mathbf{D}^{\mathrm{T}}\mathbf{w}\mathbf{D}\mathbf{d}^{-1}\mathbf{M}$, $\chi = \mathbf{M}^{\mathrm{T}}\mathbf{d}\mathbf{D}^{\mathrm{T}}\mathbf{w}$ and $\mathbf{c} = \operatorname{diag}(c_1, c_2, \ldots)$.

其中 $\boldsymbol{\mu} = \mathbf{M}^{\mathrm{T}}\mathbf{d}\mathbf{D}^{\mathrm{T}}\mathbf{w}\mathbf{D}\mathbf{d}^{-1}\mathbf{M}$, $\chi = \mathbf{M}^{\mathrm{T}}\mathbf{d}\mathbf{D}^{\mathrm{T}}\mathbf{w}$ 和 $\mathbf{c} = \operatorname{diag}(c_1, c_2, \ldots)$。

To prove the proposition, we must show that solutions (49) are incompatible with (51) for more than one component vector $\mathbf{P}$. Being lucky that we could construct solutions for P_i explicitly, we just substitute them into (51). The resulting expression is

为证明该命题，我们必须证明: 对于多于一个分量向量 $\mathbf{P}$ ，解 (49) 与 (51) 不相容。幸运的是我们可以显式构造 P_i 的解，只需将其代入 (51)，得到的表达式为

$$\mathbf{m}^2\mathbf{P} + 3H\mathbf{m}(\mathbf{P}_+ - \mathbf{P}_-) + \boldsymbol{\mu}\mathbf{P} = \chi\mathbf{c}, \tag{52}$$

where $\mathbf{P}_\pm = \operatorname{diag}\left(P_{i\pm}e^{\pm m_i t}\right)$. Each component P_i is a different exponent, and in order to satisfy the latter equation, we must put to zero coefficients in front of each of them. Moreover, we must have the constant term on the right-hand side vanish. If $H \neq 0$ we essentially must require the matrix $\mathbf{m}$ to be zero, and this is equivalent to having all $w_i = 0$ and as such we come back to the situation $\Box R = 0$ which is just a sub-case of (28) and in no way requires any more general form of R than a single Fourier mode.

其中 $\mathbf{P}_{\pm} = \mathrm{diag}\left(P_{i\pm}e^{\pm m_i t}\right)$ 。每个分量 P_i 对应不同的指数，要满足后一个方程，必须令每个指数前的系数为零，此外还必须令右侧的常数项为零。若 $H \neq 0$ ，我们本质上要求矩阵 $\mathbf{m}$ 为零，这等价于令所有 $w_i = 0$ 为零，于是我们回到情况 $\Box R = 0$ ，而这只是 (28) 的一个子情况，绝不要求 R 具有比单傅里叶模更一般的形式。

This completes the proof of the proposition during which we actually have never used that the metric must be exactly of a spatially flat FLRW type.

至此命题证明完成，在证明过程中我们实际上从未要求度量必须严格为空间平坦 FLRW 型。

In a slightly exotic situation such that there is a space-homogeneous metric which generates vanishing factor H in (50) in a combination with a nonconstant R , one needs to have a separate consideration regarding the space of solutions to EOM.

在极少数特殊情况下，若存在空间齐次度量，使得 (50) 中的因子 H 与非恒定 R 结合后等于零，则需要单独讨论运动方程的解空间。

Discussion on Classical Dynamics

经典动力学讨论

We just have proven a very important fact related to theories of type (25): all space-homogeneous conformally flat background solutions are subject to Eq. (28) in combination with conditions (30).

我们刚刚证明了一个与 (25) 类理论相关的重要事实: 所有空间齐次共形平坦背景解都满足方程 (28) 与条件 (30) 的组合约束。

To understand what happens, we must examine conditions (30) which tell that a nontrivial solution (i.e., a solution more involved than a constant R) exists only if there is a point r_1 such that function $\mathcal{F}_R(r_1)$ being considered as a function of r_1 as its parameter obeys two independent algebraic conditions. Moreover, a would-be solution must obey an equation which can be derived from a local R^2 gravity. It was elaborated in [41] what a Lagrangian for a local model must be written such that its equation of motion yields $\Box R = r_1 R$. So essentially we should worry whether function $\mathcal{F}_R(r_1)$ provides a chance to have at least some solution.

要理解具体情况，我们必须考察条件 (30)，该条件表明: 非平凡解 (即比常数 R 更复杂的解) 存在的前提是存在点 r_1 ，使得将 $\mathcal{F}_R(r_1)$ 视为以 r_1 为参数的函数时，满足两个独立的代数条件。此外，候选解还必须满足一个可从局域 R^2 引力推导得出的方程。文献 [41] 早已阐明，局域模型的拉格朗日量必须满足其运动方程给出 $\Box R = r_1 R$ 。因此从本质上说，我们需要关注的是函数 $\mathcal{F}_R(r_1)$ 是否至少能允许解存在。

The other point of view is to consider the presence of a solution as a criterion for function $\mathcal{F}_R(r_1)$ such that it provides a choice of points r_1 at which conditions (30) are true. Since functions $\mathcal{F}_X(\Box)$ are not constrained so far apart from being analytic at the origin, one may wonder about the space of solutions. Indeed, it is possible that a generic function $\mathcal{F}_R(r_1)$ has many or may be even infinitely many points in which $r_1\mathcal{F}_1$ is the same value and plus to this $\mathcal{F}_R^{(1)}(r_1) = 0$ at these points. This is some sense would mean that our model include multiple copies of local R^2 gravity.

另一种观点是，将解的存在作为函数 $\mathcal{F}_R(r_1)$ 满足条件的判据: 即存在一组点 r_1，使得条件 (30) 在这些点成立。由于除了在原点解析外，函数 $\mathcal{F}_X(\Box)$ 目前不受其他约束，人们自然会好奇解空间的情况。事实上，一般的函数 $\mathcal{F}_R(r_1)$ 很可能存在许多个，甚至可能无穷多个点，在这些点上 $r_1\mathcal{F}_1$ 取值相同，且同时满足 $\mathcal{F}_R^{(1)}(r_1) = 0$。在某种意义上这意味着我们的模型包含了局域 R^2 引力的多份拷贝。

Even though mathematically possible, we are going to remind that the operator functions $\mathcal{F}_X(\Box)$ get severely constrained by demand that no new excitations must appear in the spectrum. Indeed, as it was derived in all the details in [21], the quadratic Lagrangian (23) for the spin-0 mode of the metric around the Minkowski space-time (where we must fix the operator functions) is

尽管在数学上成立，我们仍需指出: 算符函数 $\mathcal{F}_X(\Box)$ 受到谱中不能出现新激发态这一要求的严格限制。实际上，正如文献 [21] 中已完整推导得到，围绕闵可夫斯基时空 (我们需要在此固定算符函数) 的度量自旋 0 模式对应的二次拉格朗日量 (23) 为

$$S_0 = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\phi\bar{\Box}\left\{1 - 2\frac{\lambda}{M_P^2}3\bar{\Box}\mathcal{F}\left(\bar{\Box}\right)\right\}\phi. \tag{53}$$

In order to contain the spectrum of excitations and to have inflation, we must require that the expression in curly brackets has exactly one zero which would correspond to the scalaron. This can be achieved by demanding:

为了约束激发谱并实现暴胀，我们要求大括号内的表达式恰好有一个零点，对应标量子。这可以通过以下要求实现:

$$1 - 2\frac{\lambda}{M_P^2}3\bar{\Box}\mathcal{F}\left(\bar{\Box}\right) = \sigma_0\left(\bar{\Box} - M^2\right)e^{2\sigma(\bar{\Box})}. \tag{54}$$

Here $\sigma(\Box)$ must be an entire function. For the definiteness we assume that $\sigma(0) = 0$ and in order not to lose generality we introduce σ_0. Recall that $\mathcal{F}(0) = f_0$ and evaluating left and right sides of (54) at $\bar{\Box} = 0$, we get:

此处 $\sigma(\Box)$ 必须是整函数。为明确起见，我们假设 $\sigma(0) = 0$，且为不损失一般性，我们引入 σ_0。回顾 $\mathcal{F}(0) = f_0$，将式 (54) 的左右两端在 $\bar{\Box} = 0$ 处求值，可得:

$$1 = -\sigma_0 M^2$$

yielding $\sigma_0 = -1/M^2$. Next, evaluating (54) at $\bar{\Box} = r_1$ and accounting (30) we get

得到 $\sigma_0 = -1/M^2$。接下来，将式 (54) 在 $\bar{\Box} = r_1$ 处求值，并结合式 (30)，我们得到

$$0 = -\frac{1}{M^2}\left(r_1 - M^2\right)e^{2\sigma(r_1)}.$$

This implies that $r_1 = M^2$. Differentiating (54) with respect to the d' Alembertian, evaluating the result at $\bar{\Box} = r_1$, and accounting (30), one gets:

这说明 $r_1 = M^2$ 。对达朗贝尔算符求导式 (54)，将求导结果在 $\Box = r_1$ 处求值，并结合式 (30)，可得:

$$-2\frac{\lambda}{M_P^2}3\mathcal{F}_1 = -\frac{1}{M^2}e^{2\sigma(r_1)}.$$

This implies $\sigma(r_1) = 0$.

这说明 $\sigma(r_1) = 0$ 。

The above results confirm the derivation done in [41]. However, the more important observation is that the condition $r_1 = M^2$ together with the demand that only one excitation can exist guarantees that from the point of view of the physics of our model, only a single unique point r_1 is allowed.

上述结果证实了文献 [41] 中的推导。不过更值得注意的是，条件 $r_1 = M^2$ 结合“仅能存在一个激发”的要求保证了: 从我们模型的物理视角来看，仅允许存在唯一单点 r_1 。

This is a very powerful statement because it implies that as long as space-homogeneous and conformally flat metrics are considered, our quadratic AID gravity has exactly the same space of solutions as a local R^2 gravity. In particular this means that the inflationary background will remain an attractor behavior as it was found originally for the local R^2 model in [48].

这是一个非常有力的结论，因为它表明: 只要考虑空间齐次且共形平坦的度规，我们的二次 AID 引力就和局部 R^2 引力拥有完全相同的解空间。这尤其意味着，膨胀背景仍会保持吸引子行为，正如当初在文献 [48] 的局部 R^2 模型中得到的结果。

To conclude this section, we notice that there are already mentioned limitations of our analysis: we are talking about space-homogeneous and conformally flat solutions (while allowing anisotropic metrics like of the Bianchi I types in the absence of the Weyl tensor term from the very beginning), we do not have matter sources apart from perhaps radiation. Also, we do not have the cosmological term in the action.

在本节的最后，我们指出我们的分析已经提及的局限性: 我们讨论的是空间齐次且共形平坦的解 (从一开始就允许不存在魏尔张量项的 Bianchi I 型各向异性度规)，除辐射外我们未考虑其他物质源，且我们的作用量中也不包含宇宙学项。

Models with Nonzero Cosmological Term

含非零宇宙学项的模型

In this section we extend the previous achievement to the case of a nonzero cosmological constant, the part which has never been done before.

本节我们将之前的研究成果推广至非零宇宙学常数的情形，这部分内容此前从未有人完成过。

EOM with Λ and Solution Construction

含 Λ 的运动方程与解构造

The presence of a cosmological term Λ in the model leads to several modifications of the corresponding equations. As a general point, presence of Λ in action (19) makes the Minkowski background impossible. The most easy way to see this is to write down equations of motion and check out that Minkowski background is incompatible. Namely, equation of motion now reads (see [45]):

模型中存在宇宙学项 Λ 会对对应方程带来多处修正。一般来说，作用量 (19) 中存在 Λ 会使得闵氏背景不再成立。最简便的验证方法是写出运动方程，确认闵氏背景与之不相容。具体而言，运动方程现在为 (参见文献 [45]):

$$E_{\upsilon}^{\mu} \equiv -\left(M_P^2 + 2\lambda\mathcal{F}\left(\Box\right)R\right)G_{\upsilon}^{\mu} - \frac{1}{2}\lambda\delta_{\upsilon}^{\mu}R\mathcal{F}\left(\Box\right)R + 2\lambda\left(\nabla^{\mu}\partial_{\upsilon} - \delta_{\upsilon}^{\mu}\Box\right)\mathcal{F}\left(\Box\right)R$$

$$+\lambda\mathcal{L}_{\upsilon}^{\mu} - \frac{\lambda}{2}\delta_{\upsilon}^{\mu}\left(\mathcal{L}_{\sigma}^{\sigma} + \overline{\mathcal{L}}\right) + 2\lambda\left(R_{\alpha\beta} + 2\nabla_{\alpha}\nabla_{\beta}\right)\mathcal{F}_{W}\left(\Box\right)W_{\upsilon}^{\alpha\beta\mu} + \mathcal{O}\left(W^2\right)$$

$$= \delta_{\upsilon}^{\mu}\Lambda \tag{55}$$

differing only by a nonzero constant contribution in the right-hand side. All notations are as in (26) Evidently, Minkowski background which has $R_{\mu\upsilon\rho\sigma} = 0$ may not be a solution here as long as $\Lambda \neq 0$.

其仅在右侧相差一个非零常数项。所有记号与式 (26) 一致。显然，只要 $\Lambda \neq 0$ ，含 $R_{\mu\upsilon\rho\sigma} = 0$ 的闵氏背景就不可能是此处的解。

The trace equation reads:

迹方程为:

$$E = \left(M_P^2 - 6\lambda\Box\mathcal{F}_R\left(\Box\right)\right)R - \lambda\left(\mathcal{L}_{\mu}^{\mu} + 2\overline{\mathcal{L}}\right) + \mathcal{O}\left(W^2\right) = 4\Lambda. \tag{56}$$

One can solve the above equation by modifying ansatz (28) as follows:

我们可以通过如下方式修改假设 (28)，以求解上述方程:

$$\Box R = r_1 R + r_2. \tag{57}$$

and also by adjusting conditions (30) to be

并相应将条件 (30) 调整为

$$\mathcal{F}^{(1)}\left(r_1\right) = 0, \frac{r_2}{r_1}\left(\mathcal{F}_1 - f_0\right) = -\frac{M_P^2}{2\lambda} + 3r_1\mathcal{F}_1, 4r_1\Lambda = -r_2 M_P^2 \tag{58}$$

Indeed, Eq. (56) upon substituting (57) simplifies to

将 (57) 代入式 (56) 后，确实可化简为

$$E = A_1 R - \lambda \mathcal{F}^{(1)}(r_1)\left(2r_1 R^2 + \partial_\mu R \partial^\mu R\right) + A_2 = 4\Lambda, \tag{59}$$

where

其中

$$A_1 = M_P^2 - \lambda\left(4\mathcal{F}^{(1)}(r_1)\, r_2 - 2\frac{r_2}{r_1}(\mathcal{F}_1 - f_0) + 6\mathcal{F}_1 r_1\right) \text{ and}$$

$$A_2 = -\lambda\frac{r_2}{r_1}\left(2\mathcal{F}^{(1)}(r_1)\, r_2 - 2\frac{r_2}{r_1}(\mathcal{F}_1 - f_0) + 6\mathcal{F}_1 r_1\right).$$

One can readily do some algebra to see our modified conditions work making $A_1 = 0$ and $A_2 = 4\Lambda$.

只需简单整理就能看出我们修改后的条件成立，使得 $A_1 =$ 为 0 且满足 $A_2 = 4\Lambda$ 。

Absence of Nontrivial Solutions to (56) Without Imposing (58)

不 imposing 条件 (58) 时方程 (56) 不存在非平凡解

The first question is to try again solving the latter equation as an equation without imposing conditions on A_1 and A_2 . To simplify the things, let us shift R by a constant, so far without any connection to any ansatz. We thus introduce:

第一个问题是: 我们不针对 A_1 和 A_2 施加条件，重新尝试求解后一个方程。为简化计算，我们将 R 平移一个常数，目前这一步与任何假设都无关。我们因此引入:

$$\widetilde{R} = R + \tilde{r}$$

Note that if $\tilde{r} = r_2/r_1$, then we transform (57) to the form (28) as it was without a cosmological term. Substituting $R = \widetilde{R} - \tilde{r}$ into (59) gives:

注意，若 $\tilde{r} = r_2/r_1$ ，我们可以将 (57) 变换为形式 (28)，这和没有宇宙学项的情况一致。将 $R = \widetilde{R} - \tilde{r}$ 代入 (59) 可得:

$$E = A_1\widetilde{R} - A_1\tilde{r} - \lambda\mathcal{F}^{(1)}(r_1)\left(2r_1\left(\widetilde{R} - \tilde{r}\right)^2 + \partial_\mu\widetilde{R}\partial^\mu\widetilde{R}\right) + A_2 = 4\Lambda. \tag{60}$$

Here we see that we can choose $\tilde{r}$ to eliminate all constant terms reducing the problem to the one discussed in section ”Propagator”. To be precise we get an equation:

由此可见，我们可以选取 $\tilde{r}$ 消去所有常数项，将问题约化为“传播子”一节中讨论的问题。准确来说我们得到方程:

$$-A_1\tilde{r} - 2r_1\lambda\mathcal{F}^{(1)}(r_1)\tilde{r}^2 + A_2 = 4\Lambda \tag{61}$$

One can solve for $\tilde{r}$, shift R correspondingly, and rest with an equation:

我们可以解出 $\tilde{r}$ ，对应平移 R ，最终得到方程:

$$E = \left(A_1 + 4\lambda r_1\mathcal{F}^{(1)}(r)\right)\widetilde{R} - \lambda\mathcal{F}^{(1)}(r_1)\left(2r_1\widetilde{R}^2 + \partial_\mu\widetilde{R}\partial^\mu\widetilde{R}\right) = 0. \tag{62}$$

which is essentially equation (29) with a new coefficient in front of linear in function term.

该方程本质上就是方程 (29)，仅在函数的一次项前多了一个新系数。

Reiterating section "Propagator" this means as such that there are no nontrivial solutions as long as one tries to solve the trace equation by imposing an ansatz but not imposing conditions (58). Nontrivial means $\widetilde{R} = 0$ which in terms of R means $R = \text{const}$.

回顾“传播子”一节的结论: 只要我们通过引入假设求解迹方程、但不施加条件 (58)，就不存在非平凡解。非平凡解指 $\widetilde{R} = 0$ ，对应到 R 就是 $R =$ 为常数。

Proof That Again No Solution Beyond Ansatz Are Possible

再次证明不存在超出给定拟设的解

The second question is whether solutions beyond ansatz are possible. We again substitute $R = \widetilde{R} - \tilde{r}$ but now into (56). This $\tilde{r}$ is not connected to the one used before. Then one yields:

第二个问题是是否存在超出拟设的解。我们再次将 $R = \widetilde{R} - \tilde{r}$ 代入，但这次是代入式 (56)。此处的 $\tilde{r}$ 与之前使用的无关，由此可得:

$$E = \left(M_P^2 - 6\lambda\Box\mathcal{F}_R(\Box)\right)\widetilde{R} - M_P^2\tilde{r} - \lambda\left(\mathcal{L}_\mu^\mu + 2\overline{\mathcal{L}}\right) + \mathcal{O}\left(W^2\right) = 4\Lambda. \tag{63}$$

where explicitly

其中显式给出

$$\mathcal{L}_v^\mu = \sum_{n=1}^\infty f_n \sum_{l=0}^{n-1} \partial^\mu\left(\widetilde{R} - \tilde{r}\right)^{(l)}\partial_v\left(\widetilde{R} - \tilde{r}\right)^{(n-l-1)}, \overline{\mathcal{L}} = \sum_{n=1}^\infty f_n \sum_{l=0}^{n-1} \left(\widetilde{R} - \tilde{r}\right)^{(l)}\left(\widetilde{R} - \tilde{r}\right)^{(n-l)}$$

One can introduce:

我们可以引入:

$$\widetilde{\mathcal{L}}_v^\mu = \sum_{n=1}^\infty f_n \sum_{l=0}^{n-1} \partial^\mu\widetilde{R}^{(l)}\partial_v\widetilde{R}^{(n-l-1)}, \ \widetilde{\overline{\mathcal{L}}} = \sum_{n=1}^\infty f_n \sum_{l=0}^{n-1} \widetilde{R}^{(l)}\widetilde{R}^{(n-l)}$$

and since $\tilde{r}$ is a constant, then $\widetilde{\mathcal{L}}^{\mu}_{\nu} = \mathcal{L}^{\mu}_{\nu}$. It is however may not be the case however for $\widetilde{\widetilde{\mathcal{L}}}$ as there are terms with $l = 0$. One can transform:

又由于 $\tilde{r}$ 是常数，因此 $\widetilde{\mathcal{L}}^{\mu}_{\nu} = \mathcal{L}^{\mu}_{\nu}$ 。但对于 $\widetilde{\widetilde{\mathcal{L}}}$ 情况并非如此，因为存在含 $l = 0$ 的项。我们可以做变换:

$$\overline{\mathcal{L}} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(\widetilde{R} - \tilde{r}\right)^{(l)} \left(\widetilde{R} - \tilde{r}\right)^{(n-l)}$$

$$= \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(\widetilde{R} - \tilde{r}\right)^{(l)} \widetilde{R}^{(n-l)} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \widetilde{R}^{(l)} \widetilde{R}^{(n-l)} - \tilde{r}\left(\mathcal{F}(\Box) - f_0\right)\widetilde{R}$$

$$= \widetilde{\widetilde{\mathcal{L}}} - \tilde{r}\left(\mathcal{F}(\Box) - f_0\right)\widetilde{R}$$

Now we can rewrite the trace equation as

现在我们可以将迹方程改写为

$$E = \left(M_P^2 - \tilde{r}\left(\mathcal{F}(\Box) - f_0\right) - 6\lambda\Box\mathcal{F}_R(\Box)\right)\widetilde{R} - M_P^2\tilde{r} - \lambda\left(\widetilde{\mathcal{L}}^{\mu}_{\mu} + 2\widetilde{\widetilde{\mathcal{L}}}\right) + \mathcal{O}\left(W^2\right) = 4\Lambda.$$

(64)

Choosing $\tilde{r} = -4\Lambda/M_P^2$, we eliminate the constant term. And this is indeed not the $\tilde{r}$ used for the previous question.

选取 $\tilde{r} = -4\Lambda/M_P^2$ ， 我们消去了常数项。这确实不是之前问题中使用的 $\tilde{r}$ 。

Equation to be analyzed is

待分析的方程为

$$E = \left(M_P^2 + \frac{4\Lambda}{M_P^2}\left(\mathcal{F}(\Box) - f_0\right) - 6\lambda\Box\mathcal{F}_R(\Box)\right)\widetilde{R} - \lambda\left(\widetilde{\mathcal{L}}^{\mu}_{\mu} + 2\widetilde{\widetilde{\mathcal{L}}}\right) = 0 \tag{65}$$

Here we repeat the initial steps of the previous proof without Λ . First we assume:

此处我们重复此前证明中不含 Λ 的初始步骤。首先我们假设:

$$\widetilde{R} = \sum \phi_i$$

where as before $\Box\phi_i = \omega_i\phi_i$ are Fourier harmonics. Using this one readily computes:

其中和之前一样，$\Box\phi_i = \omega_i\phi_i$ 是傅里叶谐波。由此可以直接计算得到:

$$\Box^l\widetilde{R} = \sum_i w_i^l\varphi_i,\ \mathcal{F}_R(\Box)\widetilde{R} = \sum_i \mathcal{F}_R(w_i)\varphi_i \tag{66}$$

and further

进一步可得

$$\widetilde{\mathcal{L}}_{\mu}^{\mu}=\sum_{i,j}\omega_{ij}\partial^{\mu}\varphi_i\partial_{\mu}\varphi_j,\ \widetilde{\widetilde{\mathcal{L}}}=\sum_{i,j}\omega_{ij}w_j\varphi_i\varphi_j,\ \omega_{ij}=\frac{\mathcal{F}_R(w_i)-\mathcal{F}_R(w_j)}{w_i-w_j}.$$

(67)

Notice that for $i=j$, we have to use the Taylor series expansion to obtain $\omega_{ii}=\mathcal{F}_R^{(1)}(w_i)$ where the superscript $^{(1)}$ denotes the derivative with respect to an argument. Substituting all of that into (65), one yields:

注意对于 $i=j$ ，我们需要使用泰勒展开得到 $\omega_{ii}=\mathcal{F}_R^{(1)}(w_i)$ ，其中上标 $^{(1)}$ 表示对自变量求导。将所有结果代入式 (65)，可得:

$$M_P^2\sum_k\varphi_k+\frac{4\Lambda}{M_P^2}\left(\mathcal{F}(\omega_k)-f_0\right)\varphi_k-6\lambda\sum_k w_k\mathcal{F}_R(w_k)\varphi_k-\lambda$$

$$\times\sum_{i,j}\omega_{ij}\left(\partial^{\mu}\varphi_i\partial_{\mu}\varphi_j+\left(w_i+w_j\right)\varphi_i\varphi_j\right)=0. \tag{68}$$

where the second term is new compared to the previous proof.

其中第二项是相比此前证明新增的项。

First we note that the technique of equating coefficient to zero does not work in this general case. Indeed, the quadratic in φ_i term in (68) can be eliminated by requiring $\mathcal{F}_R(w_i)=\mathcal{F}_R\left(w_j\right)$ and $\mathcal{F}_R^{(1)}(w_i)=0$ for any i,j . This being substituted into the terms linear in φ_i yields:

我们首先指出，令系数为零的方法在这个一般情况下不成立。事实上，式 (68) 中 φ_i 的二次项可通过对任意 i,j 要求 $\mathcal{F}_R(w_i)=\mathcal{F}_R\left(w_j\right)$ 和 $\mathcal{F}_R^{(1)}(w_i)=0$ 来消去。将其代入 φ_i 的一次项后可得:

$$M_P^2\sum_k\varphi_k+\frac{4\Lambda}{M_P^2}\left(\mathcal{F}(\omega_1)-f_0\right)\varphi_k-6\lambda\mathcal{F}_R(w_1)\sum_k w_k\varphi_k=0.$$

Since however different φ_k are eigenfunctions of d' Alembertian with different eigenvalues, they are linearly independent. This means that in order to satisfy the latter equation, we must require $M_P^2+\frac{4\Lambda}{M_P^2}\left(\mathcal{F}(\omega_1)-f_0\right)-6\lambda\mathcal{F}_R(w_1)w_k=0$ for each k , and as such all w_k are equal. We thus effectively come back to the situation $\widetilde{R}=\varphi_1$ like it is served by (57).

但由于不同的 φ_k 是达朗贝尔算符对应不同本征值的本征函数，它们线性无关。这意味着为满足后一个方程，我们必须对每个 k 要求 $M_P^2+\frac{4\Lambda}{M_P^2}\left(\mathcal{F}(\omega_1)-f_0\right)-6\lambda\mathcal{F}_R(w_1)w_k=0$ ，因此所有 w_k 都相等。我们实际上就回到了式 (57) 所给出的 $\widetilde{R}=\varphi_1$ 情况。

Thus, we must keep the quadratic terms in (68) and solve it as a differential equation on φ_i . Satisfying (68) will necessarily produce stringent constraints since the resulting solution for $\widetilde{R}$ must be identical to the Ricci scalar constructed from the metric. Note that in the beginning of the proof, we have mentioned that $\widetilde{R}$ is just some function of time. Here we explicitly make reference to its relation to the metric. This, however, in no way complicates the use of desired spectral properties of the d'Alembertian.

因此，我们必须保留式 (68) 中的二次项，并将其作为关于 φ_i 的微分方程求解。要满足式 (68) 必然会给出严格约束，因为得到的 $\widetilde{R}$ 解必须与从度规构造出的里奇标量完全一致。注意我们在证明开头就提到，$\widetilde{R}$ 只是时间的某个函数。此处我们明确给出了它和度规的关联。但这一点完全不会影响达朗贝尔算符预期谱性质的应用。

Going further one can pass to modified quantities $\widetilde{\varphi}_i = \varphi_i + c_i$ where we have done shifts by constants defined as

进一步来看，我们可以引入修正量 $\widetilde{\varphi}_i = \varphi_i + c_i$ ，这些修正量由我们按如下方式定义的常数平移得到

$$2\lambda \sum_j \omega_{kj} \left(w_k + w_j\right) c_j + M_P^2 + \frac{4\Lambda}{M_P^2} \left(\mathcal{F}\left(\omega_k\right) - f_0\right) - 6\lambda w_k \mathcal{F}_R\left(w_k\right) = 0 \text{ for each } k. \tag{69}$$

In fact after this point modulo obviously new expressions for c_i the proof stays identical to the previous one.

实际上，在此之后，除了 c_i 会出现明显的新表达式外，证明过程与此前完全一致

This means that we have shown that even in the case of a cosmological constant quadratic in curvature, infinite derivative gravity has the same space of solution as a pure local quadratic gravity. However, several conditions apply: it is only proven so far for conformally flat backgrounds and without any dynamical matter on the right-hand side.

这意味着我们已经证明，即使对于包含曲率二次项宇宙常数的无限导数引力，其解空间也与纯局域二次引力一致。不过该结论存在几个限制条件: 目前仅证明该结论适用于共形平坦背景，且方程右侧不存在动力学物质。

We keep a generalization of our analysis to other solutions, such that non-conformally flat backgrounds for instance, as well as other directions of an investigation regarding non-local equations in gravity models for future projects.

我们将把分析推广至其他解 (例如非共形平坦背景)，以及引力模型中非局域方程相关的其他研究方向，留待未来项目完成。

Conclusions and Outlook

结论与展望

We see that higher and infinite derivative theories open a lot of perspectives but in turn also pose a lot of challenges. One of the greatest controversies is that they obviously can improve renormalizability but generate problems of unitarity. However, gravity seems to absolutely require exactly an infinite number of derivatives. Actually here, by unitarity, one may expect that the matter sector will be non-local as well. It is not however apparent how to treat new degrees of freedom or how to get rid of them altogether. Even if one can make their classical behavior bounded, it is not clear how to show this on a generic background.

我们可以看到，高阶与无限导数理论开辟了诸多研究前景，同时也带来了诸多挑战。其中最大的争议之一在于，这类理论显然可以改善可重整性，但会引发幺正性问题。然而引力似乎绝对要求恰好存在无限多导数。实际上，就幺正性而言，我们可以预期物质部分也会是非局域的。但目前尚不明确应当如何处理新自由度，或是如何彻底消去这些自由度。即便我们可以让新自由度的经典行为有界，也无法在一般背景下证明这一结论。

Pure mathematically such models demand a construction of a Fourier transform on generic manifolds- a problem which seems to be an enormous unexplored subject. At least we want to find an answer to the following question: what are the criteria on a manifold when a Fourier transform exists and generates a complete set of basis functions? Answering this question may help understand a space of solutions of higher derivative gravity theories.

纯数学层面上，这类模型要求在一般流形上构造傅里叶变换——这一问题目前仍是一块庞大的未探索领域。至少我们需要找到以下问题的答案: 当流形上存在傅里叶变换且能生成一组完备基函数时，其满足的判据是什么？解答该问题或许有助于理解高阶引力理论的解空间。

These and many other puzzles form a lot of projects for the future study, and we hope some will be answered soon.

这些谜题以及其他诸多问题构成了大量未来研究课题，我们期待其中部分问题能早日得到解答。

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